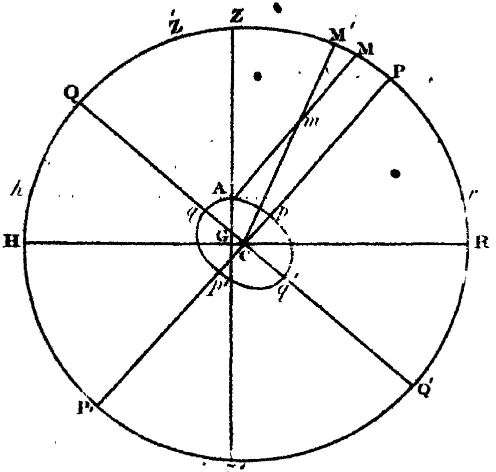
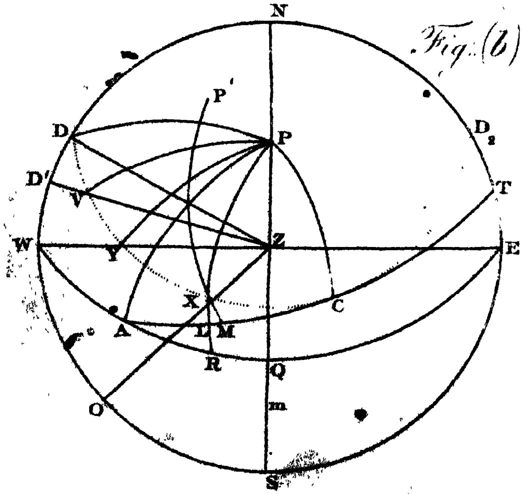


Fig: (b)



PLANE AND SPHERICAL

TRIGONOMETRY,

CONTAINING

RULES, EXAMPLES, & PROBLEMS.

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PREFACE.

The following pages contain the principal Rules in Plane and Spherical Trigonometry. The investigations of these rules requiring some knowledge of Mathematics, will be given in the second part of this treatise. The Author has been induced to make the present volume consist entirely of the rules and their applications; and to place the demonstrations in a separate part, in order that the student may be enabled to proceed through the practical part of plane and spherical trigonometry as soon as he is acquainted with vulgar and decimal fractions, and has acquired a sufficient knowledge of algebra to work an easy equation.

The collection of problems at the end of the book have been chiefly selected for Naval Students;

most of them can be solved by means of the rules contained in the book. The problems marked with the letter (*a*) have been added for the use of those who have already made some progress in mathematics; they will present little difficulty to the student who is acquainted with analytical trigonometry.

The use of the table of *log haversines** now becoming generally known, reduces considerably the labour of working out some of the problems in navigation: it may also be applied, with equal

* This table under the name of *logarithmic versed sines* may be found in MENDOZA RIOS, calculated to *five* places of decimals: the table in NORIE called *log. rising*, may be formed from it by the addition of the constant log $\bar{5}.30103$. DR. INMAN re-calculated the above table of MENDOZA RIOS, and carried it to *six* places of decimals, and arranged it in a much more convenient form for use: he has inserted it in the last edition of his tables under the name of *log. haversines*. Lastly, in the recent work on Navigation by LIEUT. RAPER, the student will find a similar table called LOG. SINE SQUARE. The reason for adopting the two last mentioned names will appear, from considering the formula, by means of which the table may be constructed from the common table of log. sines —namely, $\sin.^2\frac{A}{2} = \frac{1}{2}$ ver. A, or log. sine square $\frac{A}{2} =$ log. half versine A. RAPER derived his name from the first side of this equation: INMAN from the last, contracting *half* versine into haversine.

advantage, to the principal cases in Plane and Spherical Trigonometry. Rules have accordingly been now for the first time adapted to this table. Other rules are also given for the same cases suited to the Logarithmic Tables in more general use, such as HUTTON's, or those in MR. RIDDLE's excellent work on Navigation.

The young student who may use this volume as an Introduction to Navigation will not find it necessary to read, at first, more than a certain portion of it. The Author has marked in the table of contents with the letter (*n*) the articles which may be sufficient for this purpose. All the examples under each of the rules thus selected should be worked out; with the exception of a few of the more difficult. Such student may also solve a few of the problems at the end of the book, as far perhaps, as the 20th.

In finding the answers to the examples and problems, the Author has generally taken them from the Tables *by inspection*; that is, if INMAN's tables have been used, to the nearest 15″: if RAPER's or RIDDLE's, to the nearest minute or

half minute. It is seldom necessary in practical questions of this kind to proportion to the nearest second; and in nautical problems the above degree of accuracy will in almost every case be sufficient.

JANUARY 1, 1842.

CONTENTS.

Rules in Plane Trigonometry.

Rules in Spherical Trigonometry.

The articles marked (o) may be omitted at first by the Student.

ERRATA.

PAGE						
12	line	3	for	21872.3	read	2187.23
12	..	22	..	1507.92	.. ..	1507.82
18	*Ex.*	57	..	3	.. ..	1
18	..	58	..	4^{-6}	.. ..	4^{6}
23	..	95	add	$= c$		
28	line	4	for	$) + 1$	.. ..	$+ 1)$
36	..	last	..	10.685820	.. ..	10.685850
39	..	12	..	.272	.. ..	.275
40	..	18	..	sides	.. ..	side
41	..	7	..	*log. sin. C*	.. ..	log. *c.*
41	..	15	..	41° 0′ 30″	.. ..	47° 0′ 30″
44	..	3	..	124° 58′	.. ..	125° 58′
45	..	7	..	$\tan. \frac{1}{2}(A + B)$	..	$\tan. \frac{1}{2}(A - B)$
45	*Ex.*	152	..	*Ans.* $B = 52° 54′ 30″$	..	$C = 60° 45′$
46	line	25	for	11.713589	.. ..	11.574359
50	..	5	..	1277	.. ..	1177
50	..	5	..	33° 80′ 45″	.. ..	33° 30′ 45″
61	..	2	..	angle	.. ..	angled
63	..	18 or 7 from end	..	$\overset{-}{\cos.}\, a = \overset{+}{\cot.}\, B\, \overset{+}{\cot.}\, C$		$\overset{-}{\cos.}\, a = \overset{+}{\cot.}\, B\, \overset{-}{\cot.}\, C$
64	..	11	for	100° 23′ 15″	read	100 0′ 15″
66	*Ex.*	192	.	*Ans.* $C = 114° 40′ 45″$		$c =$ 108° 0′
67	*Ex.*	201	for	19^{2}	.. ..	$19^{\frac{2}{3}}$
69	line	4	..	*b*	.. ..	*c*
70	*Ex.*	238	..	127.7	.. ..	117.7
70	*Ex.*	238	..	21321	.. ..	213.21
71	line	2	..	1104.7	.. ..	1100.1
76	..	4 from end		(15)	.. ..	(15*a*)
78	Prob.	20	for	*BAC*	.. ..	*BAH*
80	..	29	*Ans.*	$BD = 498.7$		
84	line	9	for	(46)	read	(46*a*)
85	..	2	..	*A*	.. ..	*B*
87	Prob.	64	for	S.W. read S.S.W.	for 20° 15′	read 20′ 15′
89	..	79*a*	..	$\frac{1}{2}(b + a)$	read	$\frac{1}{4}(b + a)$
92	line	3 from end	for	*n*	.. ..	*m*
98	last line		..	(94)	.. ..	(94*a*)
99	last line		..	(101)	.. ..	(101*a*)
100	Prob.	104	for	$a = 18° 45′$	.. ..	$a = 42° 30′$
101	..	110	..	148° 36′	.. ..	147° 16′
101	..	113	..	45° 38′	.. ..	43° 38′
102	..	118	..	26 37	.. ..	26 23
106	.	138	..	2 *sin.* 2*d*	.. ..	$\frac{4}{3}$ *sin.* 2*d*
107	..	150	..	121 44	.. ..	121 47
108	..	152	..	46° 46′ N.	.. ..	41° 21′

TRIGONOMETRY.

1. ONE of the principal uses of Trigonometry is to furnish Rules for finding any part of a triangle (that is, an angle or a side), when the number of parts already known is sufficient, which is the case, in general, when three of the parts of the triangle are known.

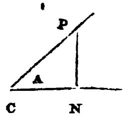
2. Considered as a branch of mathematics, Trigonometry teaches us the method of investigating certain relations of angles denominated *formulæ.* The use of these formulæ extends to every other branch of mathematics.

3. The first part of this treatise will contain the Rules for solving Plane and Spherical Triangles, with a large collection of examples and problems for practice. The problems have been selected principally to serve as an Introduction to Navigation and Nautical Astronomy.

4. In the second part, it is intended to investigate some of the more important formulæ of Trigonometry. These formulæ continually recur in Mathematics, and the student whose views extend beyond the practical part of Navigation must make himself well acquainted

with them. In the second part also, the rules given in the first part will be investigated and proved.

5. At present it will be sufficient to name only a few of the principal terms used in Trigonometry: they are as follows:



If A be any angle, and PN be drawn perpendicular to CN, then

(*a*). $\frac{PN}{CP}$ or $\frac{\text{perpendicular}}{\text{hypothenuse}}$ is called the *sine* of the angle A.

(*b*). $\frac{CP}{PN}$ or $\frac{\text{hyp.}}{\text{perp.}}$ is called the *cosecant* of angle A.

(*c*). $\frac{PN}{CN}$ or $\frac{\text{perp.}}{\text{base}}$ is called the *tangent* of angle A.

(*d*). $\frac{CN}{PN}$ or $\frac{\text{base}}{\text{perp.}}$ is called the *cotangent* of angle A.

(*e*). $\frac{CP}{CN}$ or $\frac{\text{hyp.}}{\text{base}}$ is called the *secant* of angle A.

(*f*). $\frac{CN}{CP}$ or $\frac{\text{base}}{\text{hyp.}}$ is called the *cosine* of angle A.

also $1-$cosine of A is called the *versine* of angle A.

The above fractions are called collectively, the Trigonometrical ratios.

The values of these ratios or fractions being calculated for every minute or quarter of a minute from 0° to 45°, are formed into *tables* called Tables of sines, tangents, &c., by means of which and certain other tables of numbers called *logarithms*, questions in Trigonometry and Navigation are solved.

6. Logarithms are also employed to simplify complex

and tedious arithmetical calculations: in fact, by their means, multiplication is reduced to addition, division to subtraction, the formation of powers to multiplication, and the extraction of roots to division.

7. By methods given in the second part, the logarithms of numbers have been computed and arranged in tables, so that the logarithm of any given number may be found by inspection; and conversely, if any logarithm is given, the number corresponding to it, (which, for the sake of distinction is called the *natural* number) may also be taken out.

Logarithms usually consist of a whole number and a decimal fraction: thus the logarithm of the natural number 200 is 2.301030: the whole number 2 is called the *index* or characteristic; the decimal part, namely ·301030, is called the *mantissa*. In the tables the decimal part only is inserted; the index being found by the following rules.

8. Any number being given, to find the index of its logarithm.

Rule 1.

First. When the number is an integer or whole number. The index of the logarithm is less by one, than the number of *integral* places contained in the number.

Examples.

thus the *index* of the log. of 24 is 1
.... of 2463 is 3
.... of 147.2 is 2
.... of 14.72 is 1
.... of 1.472 is 0

Rule 2.

9. **Second.** When the number is a decimal fraction. Consider the fraction as a whole number and find the index of its logarithm by the preceding rule: then subtract the number thus found from the number of decimals in the given fraction, and place over the remainder the negative sign —

Examples.

Thus, the index of log. ·123 is	$\bar{1}$
log. ·0123 is..........	$\bar{2}$
log. ·00064 is	$\bar{4}$
log. ·000000721 is	$\bar{7}$

10. If the index of the logarithm of a *vulgar* fraction be required, such fraction should be reduced, in the first place, to an equivalent decimal fraction, and the index of its logarithm found by the above rules.

Examples.

Thus, the index of log. $\frac{1}{8}$ or of log. ·125 is......	$\bar{1}$
log. $24\frac{2}{5}$ or of log. 24.4 is....	1
log. $\frac{1}{25}$ or of log. ·04 is......	$\bar{2}$
log. $\frac{1}{120}$ or of log. ·00833....	$\bar{3}$

11. Any number being given, to take from the tables its logarithm.

The rules for taking out a logarithm for any given number, and when the logarithm is given, to find the natural number corresponding to it, are contained in the explanations which accompany the table. We will here

give a collection of numbers whose logarithms are required to be found from the tables.

Examples.

Take out from the tables the logarithms of the following numbers.

			Answers.
Nat. No.	24	its logarithm is	1.380211
....	248	....	2.394452
....	2480	....	3.394452
....	1476	....	3.169086
....	14.06	....	1.147985
....	1.406	....	0.147985
....	3847	....	3.585122
....	38475	....	4.585178
....	384.75	....	2.585178
....	384.757	....	2.585186
....	4196.534	....	3.622890
....	121004.15	....	5.082800
....	$48\frac{3}{4}$ or 48.75	....	1.687975
....	$49\frac{1}{4}$ or 49.25	....	1.692406
....	2000	....	3.301030
....	200000	....	5.301030
....	$1000\frac{1}{2}$	....	3.000216
....	2.4	....	0.380211
....	·24 art (9)	....	$\bar{1}$.380211
....	.024	....	$\bar{2}$.380211
....	·000035	....	$\bar{5}$.544068
....	1415216	....	6.150822
	$\frac{3}{4}$	....	1.875061

		Answers.
Nat. No. $\frac{1}{2}$	its logarithm is	$\bar{1}.698970$
.... $\frac{1}{25}$	....	$\bar{2}.602060$
.... $\frac{1}{2000}$	....	$\bar{4}.698970$
.........004	....	$\bar{3}.602060$
.........04	....	$\bar{2}.602060$
.........75	....	$\bar{1}.875061$
.........6945	....	$\bar{1}.841672$
.... $90\frac{1}{25}$	....	1.954435
.... $1000\frac{1}{125}$	....	3.000003
.... $387\frac{1}{2}$	....	2.588272
.... $726\frac{1}{4}$	....	2.861086

12. To take out the natural number corresponding to any given logarithm.

For the method of taking out the nat. no. see explanation of tables.

To find the place of the decimal point.

FIRST. When the indices of the given logarithms are positive.

The number of integral places in the natural number will be *one* more than the index, that is, if the index of a logarithm is 2, the natural number contains three integral places; and so on: the rest of the figures in the natural number taken out are decimals.

EXAMPLES.

Required the natural numbers of the following logarithms.

Logarithms.		*Answers.*
1.380211	the Nat No. is........	24
2.394452	....	248
3.394452	....	2480
6.394452	....	2480000
2.415671	....	260.417
1.415674	....	26.042
2.310101	....	204.221
4.196171	....	15709.86
0.217845	....	1.65137
1.841569	....	69.4335
2.841989	....	695.0064
5.082800	....	121004.15
0.147985	....	1.406
0.394452	....	2.48

13. SECOND. When the index of the given logarithm is negative. The natural number in this case will be a decimal. To find the number of ciphers (if any) to be prefixed: subtract 1 from the index (without considering the negative sign) the remainder will denote the number of ciphers to be prefixed to the figures of the natural number taken from the tables: thus if the index is $\bar{4}$ prefix three ciphers; if $\bar{2}$ prefix one cipher; if $\bar{1}$, the decimal mark is placed next to the figures taken out; and so on.

EXAMPLES.

Required the natural numbers of the following logarithms.

Logarithms.

$\bar{2}$·380211	the Nat. No. is	.024
$\bar{5}$·544068	....	·000035

Logarithms.

$\bar{1}.841672$	the Nat. No. is	.6945
$\bar{7}.875061$	....	.00000075
$\bar{3}.602060$	....	.004
$\bar{2}.394452$	....	.0248

Multiplication by Logarithms.

14. Add the logarithms of the given numbers together, and the sum will be the logarithm of their product: the natural number corresponding to which is the product required.

Required the product of the following quantities.

(1). $47 \times 1{\cdot}405 \times 84$ Ans. 5546.94

(2). $\cdot 47 \times 140.5 \times \cdot 0084$ .. $\cdot 5547$

Calculation.

(1)		(2)	
log. 47 ...	1·672098	log. .47...	$\bar{1}$·672098
1.405 .	0·147676	140·5....	2·147676
84....	1·924279	.0084....	$\bar{3}$·924279
log. product	3·744053	log. product	$\bar{1}$·744053
product	5546.94	product	.5547

Examples.

Required the product of the following quantities.

(1). $72 \times 96 \times 124 \times \cdot 05$ Ans. 42854

(2). 84×96 .. 8064

(3). $6 \times 4 \times 12 \times 32$ .. 9216

(4). $64 \times 362 \times \cdot 4$ .. 9267

(5). $36 \times 48 \times 62 \times \cdot 4$ .. 42854

(6).	$1234 \times 9671 \times \cdot 00617$	Ans.	73632
(7).	$2{\cdot}4 \times \cdot 007 \times \cdot 54 \times \cdot 1$	..	·0009072
(8).	$784 \times \cdot 000079 \times \cdot 0000036$	..	·0000002229

Division by Logarithms.

15. From the logarithm of the dividend, subtract the logarithm of the divisor: the remainder will be the logarithm of the quotient: the natural number corresponding to which will be the quotient required.

1. Divide 472 by 32.2	Ans.	14.66
2. .... .0472 by 3.22	..	·01466

Calculation.

(1)		(2)	
log. 472....	2.673942	log. ·0472....	$\bar{2}$.673942
32.2 ..	1.507856	log. 3.22	0.507856
log. quotient	1.166086	log quotient ..	$\bar{2}$.166086
quotient	14·66	quotient ..	·01466

EXAMPLES.

Find the value of the following expressions.

(9).	$68 \div 34$	Ans.	2
(10).	$96 \div 1.6$	..	60
(11).	$2004.64 \div 34$	..	58.96
(12).	$19 \div 72$	..	0.26389
(13).	$19 \div .72$	..	26.389
(14).	$\dfrac{242 \times 559 \times 63}{781 \times 432}$	..	25.26
(15).	$\dfrac{84 \times \cdot 00769 \times \cdot 683}{598 \times \cdot 0000146 \times \cdot 039}$	..	1295.71

(16). $1 \div 3.2$ or $\frac{1}{3.2}$ Ans. ·3125

(17). $1 \div .45$ or $\frac{1}{.45}$ .. 2.222

(18). $1 \div \cdot 0004572$ or $\frac{1}{\cdot 0004572}$ .. 21872.3

Involution, or Raising of Powers by Logarithms.

16. FIRST. When the quantity to be raised to a power is a whole or mixed number.

Multiply the logarithm of the quantity to be raised, by the number denoting the power, and the product will be a logarithm, the natural number of which is the power required.

Required the 16th. power of 1.05 or the value of $(1.05)^{16}$.

Calculation.

$$\begin{array}{lr} \text{log. } 1.05 \ldots\ldots & 0.021189 \\ & 16 \\ \hline & 127134 \\ & 21189 \\ \hline \text{log. } (1.05)^{16} \ldots & \cdot 339024 \\ \therefore\ (1.05)^{16} \ldots = & 2.18285 \end{array}$$

EXAMPLES.

(19) Required the 6th power of 4.7215 Ans. 11078

(20) .... 3rd ...... $12\frac{1}{2}$ .. 1953.129

(21) .... 150th ...... 1.05 .. 1507·92

(22) .... 200 ...... 1.0125 .. 11.989

(23) .... 4 ...... $7\frac{4}{5}$ .. 3701·53

(24) thousandth power of 1.0125 .. 247742.3

17. SECOND. When the quantity to be raised to a power is a decimal fraction.

Multiply, separately, the index of the logarithm of the decimal fraction, and the decimal part of the logarithm by the number denoting the power.

From the former product, subtract the whole number in the latter, placing the negative sign over the remainder; and then affix to it the decimal part of the latter product, the natural number of which will be the required power; which find from the tables.

Ex. Required the tenth power of ·2

log. ·2 .... $\bar{1}.301030$

index .... $\bar{1}$ .... dec. part.. 0.301030

$$\begin{array}{r} \bar{1} \\ 10 \\ \hline \bar{10} \end{array} \qquad \begin{array}{r} 0.301030 \\ 10 \\ \hline 3.010300 \\ 10 \\ \hline \end{array}$$

log. $\cdot 2^{10}$ ...... $\bar{7}.010300$

∴ $.2^{10}$ ...... ·0000001024

(25) Required the value of the 5th power of ·2 or $(\cdot 2)^5$.. Ans. ·00032

(26) .... $(\cdot 8)^3$...... ·512

(27) .... $(\cdot 09163)^4$..000070494

(28) .... $(\cdot 975)^{200}$..·0063241

Evolution, or Extracting of Roots, by Logarithms.

18. Divide the logarithm of the given number, by the number denoting the root to be extracted, and the result will be the logarithm of the root, which find from the tables.

(1). Required the cube root of 1234
(2). Required the fifth root of ·00005214

Calculation.

(1)		(2)	
log. 1234..3)	3.091315	log. ·00005214..5)	$\bar{5}.717171$
log. $\sqrt[3]{1234}$..	$\bar{1}.030438$	log. $\sqrt[5]{\cdot 00005214}$..	$\bar{1}.143434$
∴ $\sqrt[3]{1234}$..	10.73	∴ $\sqrt[5]{.00005214}$..	.1391

EXAMPLES.

(29)	Required the 5th root of 784	Ans.	3.79195
(30)	.... square root of 365	..	19.10498
(31)	.... cube root of 12345	..	23.11162
(32)	.... 10th root of 2	..	1.071776
(33)	.... square root of .093	..	.304959
(34)	.... 5th root of 7.0825	..	1.479235
(35)	.... 365th root of 1.045	..	1.000121
(36)	.... cube root of ·00125	..	.1077

19. When the index of the logarithm to be divided by the number denoting the root is *negative*, and is not exactly divisible by that number, increase the index by as many units as will render it exactly divisible, annexing to the decimal part the same number of units so added to the index: then divide as before. Thus, to divide $\bar{4}.681241$ by 3, write it down thus; $\bar{6}+2\cdot681241$, then $\bar{6}+2\cdot681241$ divided by 3 is $\bar{2}\cdot893747$: the natural number of which is ·078297.

(1) Required the square root of .1452 or the value of $\sqrt{\cdot 1452}$

(2) Required the tenth root of .00345 or the value of $\sqrt[10]{\cdot 00345}$.

Calculation.

(1).	(2)
log. 1452 . . $\bar{1}.161967$	log. .00345 . . . . $\bar{3}.537819$
or $\bar{2} + 1.161967$	or $\overline{10} + 7.537819$
log. $\sqrt{\cdot 1452} = \bar{1}.580983$	log. $\sqrt[10]{\cdot 00345} = \bar{1}.753782$
$\therefore$ $\sqrt{\cdot 1452} = .381$	$\therefore$ $\sqrt[10]{\cdot 00345} = .5672$

EXAMPLES.

(37) Required the cube root of ·0125 *Ans.* ·2321

(38) . . square root of ·0093 . . ·09644

(39) . . . . 72nd root of ·096 . . .96797

(40) . . . . cube root of .000048. . ·036342

To find the value of a number whose index is fractional.

20. Multiply the logarithm of the number by the numerator of the fractional index, and divide the result by the denominator: the natural number corresponding to the quotient will be the value required.

(1) Required the value of $(6.025)^{\frac{4}{7}}$.

(2) . . . . . . $(.0125)^{1.5}$. or $\cdot(0125)^{\frac{3}{2}}$.

Calculation.

(1)		(2)	
log. 6.025	0.779957	log. .0125	$\bar{2}.096910$
	4		3
	7)3·119828		2)$\bar{6}$·290730

log. $(6·025)^{\frac{4}{7}} = 0·445689$ log. $(·0125)^{\frac{3}{2}} = \bar{3}·145365$

$\therefore$ $(6.025)^{\frac{4}{7}} = 2.79034$ $\therefore$ $(·0125)^{\frac{3}{2}} = ·001397$

Examples.

Required the value of the following expressions.

(41) $(·096)^{\frac{5}{9}}$ Ans. 0·272016

(42) $(19)^{\frac{4}{5}}$ .. 10·5439

(43) $\dfrac{(466871)^{\frac{6}{7}} \times \sqrt[9]{3576^{16}}}{996003 \times \sqrt{·0077}}$ .. 1788846

(44) The cube root of $(.001234)^2$ or $(.001234)^{\frac{2}{3}}$ ·01159

(45) $(472)^{\frac{5}{9}}$ 30.586

(46) $(·042)^{8·3}$ ·000000000003741

(47) $(·00563)^{·07}$ ·6958825

21. The rules contained in art 16..20 are derived from the same property of logarithms, namely, that log. $a^n = n$ log. a where n may be any number, whole or fractional: thus the log. of $6^5 = 5$ log. 6, log. $\sqrt[3]{7} = \frac{1}{3}$ log. 7; log. $·2^{\frac{3}{4}} = \frac{3}{4}$ log. ·2.

22. If the index n of the quantity be negative, it will be perhaps the easiest way to reduce it in the first place, to an equivalent expression with a positive index: this

is done by merely putting down the reciprocal of the given quantity with the sign of the power changed: thus $a^{-n} = \frac{1}{a^n}$, $x^{-3}=\frac{1}{x^3}$, $4^{-\frac{1}{3}}=\frac{1}{4^{\frac{1}{3}}}$, $\frac{1}{2^{-5}} = 2^5$: hence the log. of $a^{-n} = \log. \frac{1}{a^n} = \log. 1 - \log. a^n = 0 - n \log. a$ (since log. $1=0$.); $\log. x^{-3} = \log. \frac{1}{x^3} = \log. 1 - 3 \log. x$, $\log. 4^{-\frac{1}{3}} = \log. \frac{1}{4^{\frac{1}{3}}} = \log. 1 - \frac{1}{3} \log. 4$, $\log. \frac{1}{x^{-5}} = \log. x^5 = 5 \log. x$, and so on.

EXAMPLES.

Required the value of the following expressions.

(1) $.4^{-5}$ (2) $\frac{1}{4^{-5}}$ (3) $\frac{1}{.2^{-\frac{4}{7}}}$

Calculation.

$.4^{-5} = \frac{1}{.4^5}$

log. 1 .... 0.000000

log. $.4^5$ .. $\bar{2}.010300$

log. $.4^{-5}$.. 1.989700

$\therefore$ $.4^{-5}$.. $=97.656$

$\frac{1}{4^{-5}} = 4^5$

log. 4 .. 0.602060

5

log. $\frac{1}{4^{-5}}$.. 3.010300

$\frac{1}{4^{-5}}$.. $=1024$

$\frac{1}{.2^{-\frac{4}{7}}} = .2^{\frac{4}{7}}$

log. $.2$.. $\bar{1}.301030$

4

$\bar{3}.204120$

or $\bar{7}+4.204120$

$\therefore$ log. $.2^{\frac{4}{7}}$ $\bar{1}.600589$

$\therefore$ $.2^{\frac{4}{7}}$. $= .3987$

Examples to the preceding Rules.

Required the value of the following expressions.

(48)	3^7	..	..	Ans. 2187
(49)	$3^{\frac{1}{7}}$	..	..	1.16993
(50)	3^{-7}	..	..	.0004572
(51)	$3^{-\frac{1}{7}}$	..	..	·8548
(52)	$3^{-\frac{1}{2}}$	..	..	.5774
(53)	$(4.2)^{\frac{2}{7}}$	..	..	1.507
(54)	$(\cdot 045)^{-\frac{1}{5}}$	..	..	1.859
(55)	$(.045)^{-\frac{2}{7}}$	..	..	2.425
(56)	$\dfrac{1}{\cdot 2^{-4}}$	..	..	·0016
(57)	$\dfrac{3}{(.02)^{-\frac{4}{5}}}$	..	..	.04373
(58)	$\dfrac{3^7 \times 4^{-6}}{9^2}$	..	..	110591.3
(59)	$\dfrac{3^7 \times 4^{-9}}{3^{-2}}$	..	..	.07509
(60)	$\dfrac{4^{-5} \times 5^{-\frac{3}{5}}}{35 \times \sqrt{2}}$		..	.00000675
(61)	$\dfrac{7\sqrt{15}}{15} \times \cdot 0139 \sqrt{\dfrac{2}{11}}$			107.124
(62)	$\dfrac{33 \times 45 \times 14}{35 \times \sqrt{2}}$		.	420.022
(63)	$3^{2\frac{1}{2}}$	..	..	15.5884
(64)	$(4)^{2^3}$ and $(4)^{2^{\frac{1}{2}}}$		..	65536 and 7.103
(65)	$(50)^{\cdot 5^{10}}$	..	..	1.00383
(66)	$(6)^{\sqrt{5}}$	..	..	36·558

23. Sometimes examples occur in which several of the terms are connected by the signs + or — In such cases the value of each term thus connected must be found separately, and then applied according to its sign.

Examples.

Required the value of the following expressions.

(67) $\dfrac{(6\sqrt{47.5}+3\sqrt[3]{147\frac{1}{2}})^{\frac{3}{5}}.\ (48\frac{3}{4})^{\frac{4}{5}}}{3(1.05)^{20}-100\times{\cdot}004}$ Ans. $\dot{3}3.\dot{6}$

(68) $\dfrac{2(\frac{3}{2})^{15}-2}{\frac{3}{2}-1}$ 1747.6

(69) $\left(\dfrac{\frac{5}{9}\cdot\sqrt[18]{2}}{\frac{10}{17}\cdot\sqrt[28]{3}}+5^{\frac{4}{3}}\right)^{3}$ .. 94.794

(70) $\sqrt[5]{\tfrac{7}{3}\sqrt[4]{6}}\quad+253\sqrt[3]{\dfrac{716.5}{\sqrt{2}}}-20$ ..333.03

$19^{\frac{4}{5}}-4.54$

Calculation of Example 67.

log. 47.5 ..	1.676694	log. 1.05 ..	0.021189
			20
log. $\sqrt{47.5}$ ..	0.838347		
log. 6 ..	0.778151	log. $(1.05)^{20}$ ..	0.423780
		log. 3 ..	0.477121
log. $6\sqrt{47.5}$ ..	1.616498		
∴ $6\sqrt{47.5}$ ..	= 41.35	log. $3(1.05)^{20}$.	0.900901

$\log. 147.5 \quad .. \quad 2.168792$

$\log. \sqrt[3]{147.5} \quad .. \quad 0.722931$

$\log. 3 \quad .. \quad 0.477121$

$\log. 3\sqrt[3]{147.5} \quad .. \quad 1.200052$

$\therefore 3\sqrt[3]{147.5} \quad .. \quad = 15.85$

$\therefore 6\sqrt{47.5}+3\sqrt[3]{147.5}=57.20$

$\log. 57.2 \quad .. \quad 1.757396$

3

$5)5.272188$

$\log. (57.2)^{\frac{3}{5}} \quad . \quad 1.054437$

$\log. 48\tfrac{3}{4} \quad . \quad 1.687975$

4

$5)6.751900$

$\log. (48\tfrac{3}{4})^{\frac{4}{5}} \quad .. \quad 1.350380$

$\log. (57.2)^{\frac{3}{5}} \quad .. \quad 1.054437$

$\log.$ Numerator 2.404817

$\therefore 3(1.05)^{20} \quad . \quad = 7.96$

$100 \times .004 \quad = 0.4$

$\therefore$ denominator $. = 7.56$

$\log.$ numerator .. 2.404817

$\log.$ denominator 0.878522

$\log.$ of fraction 1.526295

$\therefore$ fraction .. $= 33.6$

Of four proportional quantities any three being given to find the fourth.

24. If the required quantity be an extreme term, add together the logarithms of the two middle terms, and subtract the logarithm of the other extreme.

If the required quantity be one of the middle terms, add the logarithm of the extremes and subtract the logarithm of the other middle term.

The student will find it convenient in working proportions by logarithms to make a dash with the pen under the term required.

EXAMPLES.

Find the value of x in the following proportions.

(71) $24 : 35 :: 79 : x$ Ans. $x = 115.208$

(72) $3505 : x :: 1507 : 29.8$ $x = 69.3$

(73) Find a fourth proportional to ·0963, ·24958 and ·008967 Ans. ·02317

(74) $\sqrt{724} : \sqrt{\frac{5}{13}} :: 6.927 : x$ .. ·1589

To reduce Algebraical formulæ to logarithms.

RULE

25. (*a*) The logarithm of the product of any number of terms is equal to the sum of the logarithms of all the terms in the product: thus if $x=ab$ then $\log. x=\log. a + \log. b$.

(*b*) The logarithm of the quotient of any two numbers is equal to the logarithm of the dividend diminished by the logarithm of the divisor: thus if $x = \frac{a}{b}$ then $\log. x = \log. a - \log. b$. If $x = \frac{abc}{de}$ then $\log. x=\log. a+\log. b + \log. c - \log. d - \log. e$.

(*c*) The logarithm of the power of any quantity is equal to the logarithm of the quantity multiplied by the number denoting the power: thus if $x = a^{10}$ then $\log. x = 10 \log. a$: if $x = a^2b^3$ then $\log. x = 2 \log. a + 3 \log. b$.

(*d*) The logarithm of the root of any quantity is equal to the logarithm of the quantity divided by the number denoting the root to be extracted: thus if $x = \sqrt[3]{a}$ or $a^{\frac{1}{3}}$, then $\log. x = \frac{\log. a}{3}$ or $\frac{1}{3} \log. a$; if $x = a^{\frac{2}{5}} b^{\frac{1}{3}}$, then

$\log. x = \frac{2}{5} \log. a + \frac{1}{3} \log. b.$

Examples of reducing Algebraical expressions to logarithms.

Reduce the following expressions to logarithms.

(75) $x = abcd$ Ans. $\log. x = \log. a + \log. b + \log. c + \log. d$

(76) $x = \frac{ab}{c}$ .. $\log. x = \log. a + \log. b - \log. c$

(77) $x = \frac{ab}{cd}$ .. $\log. x = \log. a + \log. b - \log. c - \log d$

(78) $x = a^2bcd^2$ .. $\log. x = 2 \log. a + \log. b + \log. c + 2 \log. d$

(79) $x = \frac{a^2b\sqrt[3]{c}}{10}$ .. $\log. x = 2 \log. a + \log. b + \frac{1}{3} \log. c - 1$

(80) $x = \frac{\sqrt{a}.b\sqrt[4]{c}}{d}$ .. $\log. x = \frac{1}{2} \log. a + \log. b + \frac{1}{4} \log. c - \log. d$

(81) $x = \frac{\sqrt{ab}}{cd^2}$ .. $\log. x = \frac{1}{2} \log. a + \frac{1}{2} \log. b - \log. c - 2 \log. d$

(82) $x = \frac{\sqrt[3]{abc^4}}{\sqrt[4]{d}}$. $\log. x = \frac{1}{3} \log. a + \frac{1}{3} \log. b + \frac{4}{3} \log. c - \frac{1}{4} \log. d$

26. By means of the preceding rules, the following equations may be solved.

Find the value of x in the following equations.

(83) $10^x=456$ .. .. Ans. $x=2.658965$

(84) $x^3=14$ .. .. $x=2.4101$

(85) $x^5=14.76$ .. .. $x=1.71323$

(86) $3^{2x}=20$ .. .. $x=1.3634$

(87) $x^3=.004$ .. .. $x=.1587$

(88) $x^{-3}=4\frac{1}{2}$ .. .. $x=.6057$

(89) $x=(.02445)^{\frac{3}{4}}$ .. .. $x=.06183$

(90) $x=\sqrt[2000]{47691}$ .. .. $x=1.0054$

(91) $x=\sqrt{\frac{2}{123}}$ .. .. $x=.1275$

(92) $x=\sqrt{\frac{1}{3.14159}}$ .. $x=.6827$

(93) $a^x=b$ .. .. $x=\frac{\log. b}{\log. a}$

(94) $\frac{a^x}{b^x}=c$ .. .. $x=\frac{\log. c}{\log. a-\log b}$

(95) $\frac{a^{mx}}{b^{nx-1}}$ .. $x=\frac{\log. c-\log. b}{m\log. a-n\log. b}$

(96) $a^x=\frac{b^{mx-n}}{c^{rx}}$ .. $x=\frac{n\log b}{m\log b-\log. a-r\log. c}$

(97) $\left(\frac{5}{4}\right)^x=54\frac{1}{2}$ .. .. $x=17.91$

(98) $a^{\frac{b}{x}}=c$ .. .. $x=\frac{b\log. a}{\log. c}$

(99) $a^{b^x}=c$ .. .. $x=\frac{1}{\log. b}.\log.\left(\frac{\log. c}{\log. a}\right)$

(100) $(10\frac{1}{2})^x = 20$ . . Ans. $x = \dfrac{1301030}{1021189}$

(101) $2^{3^x} = 4$ .. .. $x = \dfrac{301030}{477121}$

24. Logarithms were originally invented to facilitate trigonometrical calculations. The following examples will shew their use in applications of another kind.

In Geometrical Progression. Let a = the first term of a geometrical series; r = common ratio, n = number of terms, and S = sum of terms

$$\text{then } S = a.\frac{r^n - 1}{r - 1}.$$

(102) Find the sum of 20 terms of the series,

$$1,\ \tfrac{3}{2},\ \tfrac{9}{4},\ \tfrac{27}{8}\ .$$

$$S = a.\frac{r^n - 1}{r - 1} = 1.\frac{(\frac{3}{2})^{20} - 1}{\frac{3}{2} - 1} = \frac{(1.5)^{20} - 1}{\frac{1}{2}}$$

and $\log.(1.5)^{20} = 20 \log. 1.5 = 3.521820. (1.5)^{20} = 3325.21$

$$\therefore S = \frac{3325.21 - 1}{\frac{1}{2}} = \frac{3324.21}{\frac{1}{2}} = 6648.42 \text{ Ans.}$$

(103) The sum of a geometrical series is 6560, its first term 2 and common ratio 3: find the number of terms.

$$S = a\frac{r^n - 1}{r - 1} \therefore 6560 = 2.\frac{3^n - 1}{3 - 1} = 3^n - 1.$$

$$\therefore 3^n = 6561 \text{ and } n \log. 3 = \log. 6561 \therefore n = \frac{\log. 6561}{\log. 3} = 8.$$

(104) If sum of series = 1023, first term = 1, common ratio = 2: what is the number of terms.

$$S = a\frac{r^n - 1}{r - 1} \quad \text{Ans. } n = 10$$

Without the use of logarithms the last two questions would be very difficult to answer.

In Compound Interest.—If P denote the principal, r the interest of £1 for 1 year, n the number of years, and A the amount, then $A = P(1+r)^n$: any three of these quantities being given, the fourth may be found.

(105) If £200 be placed out at compound interest for 7 years at 4 per cent: required the amount.

$A = P(1+r)^n$ ∴ $\log. A = \log. P + n \log. (1+r) =$ $\log. 200 + 7 \log. (1 + \cdot 04)$ Ans. $A =$ £263 3s. 8d.

(106) At what rate of interest must £400 be placed out that it may amount to £569 6s. 8d. in nine years, at compound interest.

$$A = P(1+r)^n \therefore \log. (1+r) = \frac{\log A - \log. P}{n}$$

Ans. $r = \cdot 04$ or 4 per cent.

(107) In how many years will £500 amount to £900 at 5 per cent, compound interest.

$$A = P(1+r)^n \therefore n = \frac{\log. A - \log. P}{\log. (1+r)} \text{ Ans. } n = 12.04 \text{ years.}$$

(108) In Astronomy it is proved that the *squares* of the times which two planets employ to make their revolutions about the sun are to each other as the *cubes* of their distances from the same heavenly body. Knowing that the revolution of the earth about the sun is performed in 365 days, 5 hours, 48 minutes, 51 seconds, and that of Jupiter in 4330 days, 14 hours, 39 minutes, 2 seconds: it is required to find the ratio of the distances of these planets from the sun.

Reducing in the first place into seconds the given re-

volutions, we have 31556931 seconds for the earth; and 374164742 for Jupiter. Representing the first by a, the second by b, the distance of the earth from the sun by 1, and that of Jupiter by x, we have

$$x^3 : 1^3 :: b^2 : a^2$$

$$\text{or } x^3 = \frac{b^2}{a^2}.$$

$\therefore$ 3 log. $x = 2$ log $b - 2$ log. a and log. $x =$

$$\frac{2 \log b - 2 \log a}{3} = 0{\cdot}715978 \;\therefore\; x = 5{\cdot}1997.$$

Or the distance of Jupiter from the sun is to that of the earth from the same body, nearly as 52 . 10.

In Gunnery.— If r denote the resistance of the air (in lbs.) d the diameter of the ball, v its velocity (supposed to be less than 400 feet), then the resistance of the air is represented by the expression $(.0021\,dv)^2$.

(109) Required the resistance (r) to a 10 inch shot, supposing it to be moving with a velocity of 250 feet in a second.

$$r = (.0021\,dv)^2$$

In logs. log. $r = 2$ (log. .0021 + log. d + log. v)

log. .0021	$\bar{3}$.322219
log. 10 .	1.000000
log. 250 ..	2.397940
	0.720159
	2
log. r	1.440318
$\therefore$ r	27.56lbs.

When the velocity of the body is greater than 400 feet the expression for r is $(\cdot 002784dv)^2 - (\cdot 04183d)^2 v$.

Ex. (110) Required the resistance (r) to a 4 inchball moving with a velocity of 1600 feet in a second.

$$r = (\cdot 002784dv)^2 - (\cdot 04183d)^2 v$$

log. 1st part $= 2$ (log. ·002784 + log. d + log. v)
log. 2nd part $= 2$ (log. ·04183 + log. d) + log. v

log. ·002784.. $\bar{3}.444669$	log. ·04183.. $\bar{2}$·621488
log. 4 ...... 0.602060	log. 4 ...... 0.602060
log. 1600 .... 3.204120	
	$\bar{1}.223548$
1·250849	2
2	
	$\bar{2}.447096$
log. 1st part.. 2.501698	log. 1600.... 3.204120
1st part...... 317.4	
	log. 2nd part 1.651216
	2nd part 44.8

$\therefore r = 317.4 - 44.8 = 272.6.$

Ex. (111) Required the terminal or greatest velocity which a shell of 12.75 inches diameter and weighing 200lbs., can acquire by falling through the air.

The expression for v, the greatest velocity is

$$\frac{1}{\cdot 0021d}\sqrt{\text{weight}}$$

In logs. Log. $v = \frac{1}{2}$ log. wt. − (log. ·0021 + log. d.)

log. 200.... 2.301030	log. .0021.... $\bar{3}.322219$
$\frac{1}{2}$ log. 200 .. 1.150515	log. 12.75.... 1.105510
$\bar{2}.427729$	$\bar{2}.427729$
log. v 2.722786	

$\therefore$ Velocity........ 528.2.

Ex. (112) Required the height to which a 4-inch ball will ascend in air, projected with a velocity of 1600 feet in a second.

The expression for height h is $669d \times \log.\left(\frac{v^2 - 231\cdot5v}{18600d}\right) + 1$

The value of fraction $\frac{v^2 - 231.5v}{18600d} + 1$ is 30.43, the log. of which is 1.483.

Then $\log. h = \log. 669 + \log. d + \log. 1.483.$

log. 669	2.825426
log. 4	0.602060
log. 1.483	0.171141
log. h	3.598627
∴ h	3969 feet.

Tables used in Trigonometry and Navigation.

28. To take out quantities from the Tables of log. sines, tangents, &c.,—natural versines, &c.

The rules for taking out quantities from these tables, are given in the explanation of the use of the tables, to which the student is referred.

EXAMPLES.

(113) Required the log. sine, log. tangent and log. cosecant of the following angles (to the nearest 15″).

				log. sine	*log. tan.*	*log. cosec.*
2°	2′	6″	Ans.	8.549995	8.550268	11.450005
19	10	40		9.516566	9.541366	10.483434
48	35	30		9.875070	10.054592	10.124930
41	24	30		9.820478	9.945408	10.179522

(114) Required the angles (to the nearest 15″) whose log. sines are

9.641452	Ans.	25°	58′	30″
9.714152	..	31	11	0
9.984204	..	74	38	30

29. When the angle is greater than 90°, subtract it from 180°, and look for the remainder, which is called its *supplement*, in the tables: thus, to find the log. sine of 100° 10′ subtract it from 180° and look for the log. sine of the remainder, (namely 79° 50′) which is 9.993127 or log. sine 100° 10′=9.993127.

30. But the readiest way, will in general be to *diminish* the given angle by 90° and to look out the remainder according to the following rule.

If A denote any angle less than 90° then,

for sine $(90+A)$ take out ...... *co*sine A
tangent $(90+A)$ .... *co*tang. A
secant $(90+A)$............ *co*sec. A
*co*sine $(90+A)$ .... sine A
*co*tang.$(90+A)$............ tan. A
*co*secant $(90+A)$ .... secant A

Thus to find log. cosine 100° or log. cosine (90+10) take out the log. sine 10°, which is .. 9.239670.

To find log. cosec. 170° 14′ 15″ take out log. sec. 80° 14′ 15″ which is .. 10.770665.

31. *Table of Natural Versines.*

EXAMPLES.

(115) Find the natural versines of the following angles.

26°	32′	15″	Ans.	105357
157	48	50	..	1925964
90	7	15	..	1002109
125	0	30	..	1573695

(116) Required the angles whose natural versines are

1175443	Ans.	100°	6′	16″
105357	..	26	32	15
1925964	..	157	48	50
1573695	..	125	0	30

The remaining tables used in Trigonometry and Navigation require no examples to explain their use.

32. To reduce Trigonometrical formulæ to tabular logarithms.

Trigonometrical formulæ are reduced to logarithms by the common rules for reducing algebraical formulæ (25); but in order to avoid as much as possible, the inconvenience of using negative indices in calculations, the table of log. sines, cosines, &c. is constructed by adding 10 to the indices of the log. sines, &c. Thus the sine 30° being equal to $\frac{1}{2}$, its logarithm is $\bar{1}.698970$, but the log. sine of 30° contained in the tables is $10+\bar{1}.698970$ or 9.698970. The tabular log. sine is therefore equal to the log. sine $+10$ and log. sine of any angle is $=$ tab. log. sine $-$ 10.

For a similar reason the table of natural versines is

constructed by multiplying the natural versines by one million: thus versine $60° = \frac{1}{2} = .5$, but the versine of $60°$ contained in the table is $= .5 \times 1000000 = 500000$ or tab. vers. $=$ ver. $\times$ 1000000 therefore log. tab. vers. $=$ log. vers. $+ 6$ or log. vers. $=$ log. tab. vers. $- 6$.

In reducing therefore a trigonometrical formula to tabular logarithms we must subtract 6 from each log. versine, and 10 from each of the other trigonometrical terms that occurs in the formula.

(117) Thus, if tan. $A =$ sin. A sec. A. In tabular logarithms, tab. log. tan. $A - 10 =$ tab. log. sin. $A - 10$ $+$ tab. log. sec. $A - 10$ or, as it is usually written (suppressing the word tabular, it being understood that the formula is reduced to tabular logarithms) log. tan. $A - 10 =$ log. sin. $A - 10 +$ log. sec. $A - 10$ and collecting the tens.

$$\text{log. tan. } A = \text{log sin. } A + \text{log. sec. } A - 10.$$

If tan. $A = \dfrac{\text{sin. } A}{\text{cos. } A}$. In tab. logs.; log. tan. $A - 10$ $=$ log. sin. $A - 10 -$ (log. cos. $A - 10$) or log. tan. A $=$ log. sin. $A + 10 -$ log. cos. A.

(118) If ver. $x = 2$ sin. b sin. c sin.$^2 \dfrac{A}{2}$. In tab. logs.; log. ver. $x - 6 = \cdot 301030 +$ log. sin. $b - 10 +$ log. sin. $c - 10 + 2$ log. sin. $\dfrac{A}{2} - 20$ or log. ver. $x =$ $6.301030 +$ log. sin. $b +$ log. sin. $c + 2$ log. sin. $\dfrac{A}{2} - 40$.

EXAMPLES.

Reduce to tabular logarithms the following formulæ.

(119) Sin. x=cosec. y tan. z. Ans. log. sin. x= log. cosec y + log. tan. z — 10.

(120) $a=b$ tan. A. Ans. log. a = log. b + log. tan. A — 10.

(121) Tan. $A=\frac{a}{b}$. Ans. log. tan. A= log. a+10 —log. b.

(122) $\frac{x}{a}$ = cot A. Ans. log. x=log. cot. A + log. a — 10. If a = 20 and A=30° 10′ then x may be thus found.

Calculation.

log. cot. A	10.235648
log. a.	1.301030
	11.536678
subtract. .	10
∴ log. x	1.536678
and x. . . .	34.41

33. Use of the algebraic signs + and — to determine the magnitude of an angle.

The numerical value of the trigonometrical ratio (art 5) of an angle and of its supplement (26) is the same, and therefore the log. of any trigonometrical ratio taken out of the tables corresponds to both angles; thus, the log. sine of 100°=log. sine of 80°. For this reason, when we have the log. sine of an angle given to find the angle, we are not certain whether the angle contained in the tables, or its supplement may be the correct one. The uncertainty however can be removed when the given

trigonometrical ratio is a tangent, cotangent, secant or cosine, by finding in the first place, whether its algebraic sign is + or —. In the former case the angle taken out of the tables is the one sought: in the latter we must subtract the angle taken out from 180° for the required angle.

34. The rule for finding the algebraic sign of a trigonometrical ratio in any equation or formula where all the other terms are known is the following.

Having written down the formula and simplified it by clearing it of fractions, &c., put over each given term its proper algebraical sign; that is, over each tangent, secant, cosine and cotangent of an angle less than 90° the sign +, and over each of the same quantities when the angle is greater than 90° the sign —: but over each sine, cosecant and versine the sign + whether the angle is greater or less than 90°, and determine from thence the sign of the product of that side whose terms are all known.

Then, since the sign of the product of each side of the equation must be the same (otherwise we should have a positive quantity equal to a negative quantity): make it so by putting over the unknown term the sign + or — accordingly.

If + falls over the unknown term, the part required is less than 90°, and the quantity taken out will be the angle; but if —, subtract the angle taken out of the tables from 180°, the remainder will then be the angle required. When the part sought is expressed in terms of the sine, the above rule will not apply since the sine is

positive, both when the angle is greater or less than 90°. The uncertainty which thence arises can only be removed in particular cases.

In the above rule the angle required is supposed to be always less than 180°.

Examples.

In the following examples it is required to find whether the angle x is greater or less than 90°: supposing $A=45°$, $B=120°$ and $C=130°$.

(123) Cos. x = tan. A cos. B sec. C.

Placing over the given terms the proper signs we have

$$\cos. x = \overset{+}{\tan.} A \ \overset{-}{\cos.} B \ \overset{-}{\sec.} C,$$

and since the product of the three terms on the right hand side of the equation is positive, cos. x must also be positive, and therefore x is less than 90°.

Calculation.

$$\overset{+}{\cos.} x = \overset{+}{\tan.} A \ \overset{-}{\cos.} B \ \overset{-}{\sec.} C$$

log. tan. A ... 10.000000
.. cos. B.... 9.698970
.. sec. C.... 10.191933

log. cos. x.... 9.890903 (rejecting 20)*
$\therefore$ x.... = 38° 56′

* By turning the formula into logarithms, we shall see that 10 must be subtracted from the left hand side, and 30 from the right; or the left hand side of the equation must be diminished by 20: thus

log.cos.x − 10 = log.tan. A − 10 + log.cos. B − 10 + log.sec.C − 10,
or log. cos. x = log. tan. A + log. cos. B + log. sec. C − 20.

In the same manner the rejection or addition of the tens in any logarithmic formula may be explained.

(124) *sec.* $x =$ sin. A sec. B: determining by the rule the signs of sin. A and sec. B, we find that *sec.* x $=$ sin. $\overset{+}{A}$, sec. $\overset{-}{B}$, a negative product, therefore sec. x is negative, or x is greater than 90°.

Calculation.

log. sin. A....	9.849485
.. sec. B....	0.301030
log. sec. x....	10.150515
	45°
	180
$\therefore\ x =$	135

(125) *Tan.* $\overset{-}{x} =$ cosec $\overset{+}{A}$ cos. $\overset{-}{B}$ placing over A and B their proper signs we find that *tan.* x is negative,

and $\therefore$ x is greater than 90°.

(126) cosec $\overset{+}{A}$ sin. $\overset{+}{B}$ $=$ *cot.* $\overset{+}{x}$

$\therefore$ x is less than 90°.

(127) sec. $\overset{+}{A}$ sin.$^2\overset{+}{C}=$cos.$^2\overset{+}{B}$ *cot.* $\overset{+}{x}$

$\therefore$ x is less than 90°.

(128) sec. A sin.$^2C = -$ cos $^2 B$ *cot.* x: placing the signs and taking into consideration the negative sign in front of the right hand side of the equation, we have,

$$\text{sec. } \overset{+}{A} \text{ sin}^2\overset{+}{C} = -\text{cos.}^2\overset{+}{B}\ \text{cot. } \overset{-}{x}$$

$\therefore$ x is greater than 90°.

(129) Sin. $\overset{+}{A}$ *cos.* $\overset{-}{x} = -$ cos. $\overset{-}{B}$ cot. $\overset{-}{C}$

$\therefore$ x is greater than 90°.

(130) Ver. $A = \dfrac{\text{cos. } B}{\text{sin. } C \text{ tan. } x}$

or ver. $\overset{+}{A}$ sin. $\overset{+}{C}$ *tan.* $\overset{-}{x} =$ cos. $\overset{-}{B}$

or x is greater than 90°.

RULES

IN PLANE AND SPHERICAL TRIGONOMETRY.

I.

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference: under which put the third side: take the sum and difference, and also the half sum and half difference.

To the arithmetical complements* of the logarithms of the two first terms in this form, add the logarithms of the two last, and reject 10 in the index: the result will be the log. haversine (*a*) of the required angle, which find in the table.

(*a*) If the student have no table of haversines the angle may be found by the following rule.

* 10 – log. *b* is called the arithmetical complement of log. *b* In practice it is easily found by taking each figure of the logarithm from 9, except the last, and that from 10: thus, ar. co log of 2.714152 is 7.285848; ar. co log. 1.314150 is 10.685820.

II.

Three sides of a plane triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference: under which put the third side: take the sum and difference, and also the half sum and half difference.

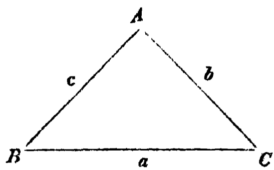
To the arithmetical complements of the logarithms of the two first terms in this form, add the logarithms of the two last, divide by 2, and look out the result as a log. sine; the angle corresponding to which will be half the required angle.

Examples.

(1) In the plane triangle ABC, given $a=20$, $b=30$, and $c=40$; required A.

(2) Given $a=\cdot025$, $b=\frac{1}{8}=\cdot125$, $c=\cdot115$. required B.

By Rule I. (using haversines.)



(1)			(2)		
30 ..	ar. co. log. ..	8.522879	·025 ..	ar. co. log. ...	11.602060
40 ..	ar. co. log. ..	8.397940	·115 ..	ar. co. log. ...	10.939302
10	log. 15 ..	1.176091	·090	log. ·1075 ..	$\bar{1}.031408$
20	log. 5 ..	0.698970	·125	log. ·0175 ..	$\bar{2}.243038$
30		———	·215		———
10	log. hav. A ..	8.795880	·035	log. hav. B ..	9.815808
15			·1075		
5	$\therefore A$ .. =	28° 57′ 15″	·0175	$\therefore B$ .. =	107° 58′ 45″

By Rule II. (without haversines.)

(3) Given $a = 512$, $b = 627$, $c = 430$: required C.

(3)

512	ar. co. log.	7.290730
627	ar. co. log.	7.202732
115	log. 272.5	2.435366
430	log. 157.5	2.197281
545		2)19.126109
315		
	log. sin. $\frac{1}{2}$ C ...	9.563054
272.5	∴ $\frac{1}{2}$ C	21° 26′ 45
157.5		2
	and C .. =	42 53 30

In the above examples the angles A B and C have been taken out of the tables by inspection, namely, to the nearest 15″. This degree of accuracy will be found in most cases sufficient. Rules for proportioning for seconds are given in the explanation of the tables.

Examples.

Required the angles of the plane triangle ABC whose three sides a, b, c, are

(131)	$a=798$	Ans. $A=$	89°	45′	37″
	$b=460$	$B=$	35	12	7
	$c=654$	$C=$	55	2	16
(132)	$a=512$	Ans. $A=$	54	8	11
	$b=627$	$B=$	82	58	11
	$c=430$	$C=$	42	53	38

(133)	$a=649$	Ans. $A=$	56°	4′	6″
	$b=586$	$B=$	48	31	4
	$c=757$	$C=$	75	24	50
(134)	$a=627$	Ans. $A=$	29	44	2
	$b=1140$	$B=$	115	36	32
	$c=718.9$	$C=$	34	39	26
(135)	$a=.025$	Ans. $A=$	10	58	0
	$b=\frac{1}{8}$	$B=$	107	58	50
	$c=.115$	$C=$	61	3	10
(136)	$a=.8$	Ans. $A=$	107	46	6
	$b=.672$	$B=$	53	7	24
	$c=.272$	$C=$	19	6	30
(137)	$a=\frac{1}{2}$	Ans. $A=$	28	14	14
	$b=\frac{3}{4}$	$B=$	45	12	34
	$c=1.013$	$C=$	106	33	12
(138)	$a=\frac{1}{4}$	Ans. $A=$	20	11	24
	$b=.541$	$B=$	48	19	11
	$c=.674$	$C=$	111	29	23

III.

Of two sides and two opposite angles, any three being given to find the fourth.

Write down a proportion, having for the two first terms the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the sine of the angle opposite the side in the second term of the proportion: mark the term required with a stroke of the pen underneath.

If the term marked be a *middle* term; add the logarithms of the two extreme terms and subtract the logarithm of the middle term not marked.

If the term marked be an *extreme* term; add the logarithms of the two middle terms, and subtract the logarithm of the extreme term.

The result will be the logarithm of the required term.

Note 1. Since the sine of an angle has the same numerical value as the sine of its supplement; if one of the angles be greater than 90° subtract it from 180° and look out the log. sine of the remainder in the table of log. sines, as the log. sine of the angle in question.

2. When two angles are known, the third can be found by adding together the two known angles and subtracting the result from 180°: the remainder will be the third angle.

3. Therefore if two angles and the adjacent sides be given, the third angle is also known, and thence by the above rule the other two sides.

4. This rule depends on the property of triangles that the sides are to one another in the same proportion as the *sines* of the angles opposite to them; and it will be, in general the easiest way of solving the example to write down such proportion, and find the unknown term by Art. (24, p. 20).

Ex. Given $a = 50\frac{1}{4}$, $b = 25$, and $A = 68° 48'$: to find the other parts. (fig. 1).

To find B. (fig. 1′)	To find C.	To find c. (fig. 1″)
$a : b :: \sin. A : \sin. B$	B.. 27° 38′ 15″	$c : a :: \sin. C : \sin. A$
log. b.... 1.397940	A.. 68 48 0	log. a 1.701136
log. sin. A 9.969567	96 26 15	log. sin. C.. 9.997253
11.367507	180	11.698389
log. a.... 1.701136	C.. 83 33 45 (Note 2)	log. sin. A.. 9.969567
log. sin. B 9.666371		log. sin. C.. 1.728822
B.... 27° 38′ 15		c .. 53.56

EXAMPLES.

Find the other parts of the triangle ABC; having given

(139)	$a=214$	Ans... $B=36°\ 6'\ 30''$
	$b=191$	$C=102\ 34\ 15$
	$A=41°\ 19'\ 15''$	$c=316.3$
(140)	$a=17.25$	Ans... $C=27°\ 7'\ 15''$
	$c=10\frac{3}{4}$ or 10.75	$B=105\ 52\ 15$
	$A=41°\ 0'\ 30''$	$b=22.69$
(141)	$a=96$	Ans... $b=73.98$
	$c=48$	$B=49°\ 0'$
(Note 1)	$A=101°\ 41'$	$C=29°\ 19'$
(142)	$c=376$	Ans... $C=91°\ 43'$
	$A=48°\ 3'$	$a=279\ 7$
(Note 3)	$B=40\ 14$	$b=243.0$
(143)	$A=60°$	Ans... $C=48°\ 0'$
	$B=72$	$c=207.6$
	$a=242$	$b=265.76$
(144)	$a=2\frac{3}{4}$	Ans... $C=88°\ 2'\ 40''$
	$A=43°\ 24'\ 10''$	$b=3$
	$B=48\ 33\ 10$	$c=4$

(145) $b=\frac{3}{4}$ Ans... $A=28° 13'$
$a=\frac{1}{2}$ $C=106\ 37$
$B=45° 10'$ $c=1.013$

(146) $A=20° 10$ Ans... $C=111° 35'$
$B=48\ 15$ $b=\cdot 541$
$a=\frac{1}{4}$ $c=\cdot 6743$

(147) $a=\cdot 02$ Ans... $B=90°$
$A=11° 32' 15''$ $c=\cdot 09797$
$C=78\ 27\ 45$ $b=\cdot 1$

(148) $A=22° 20'$ Ans... $C=108° 12' 30''$
$B=49\ 27\ 30$ $a=1$
$c=2\frac{1}{2}$ $b=2$

Ambiguous Case.

35. When two sides and the angle opposite the *less* side are given, each of the quantities sought will have two distinct values. For suppose that in the triangle ABC, (fig. 2) the sides BC and AC and the angle A opposite to the *less* side are given, then if BC be not perpendicular to AB, and from C as a center with radius CB an arc be described, it will cut AB in another point B′: then if CB′ be joined, it is manifest that there will be *two* triangles ACB and ACB′ having the given parts, namely the sides AC CB or CB′ and the angle A the same in both: while the remaining parts are different.

The angle ABC or $AB_{,}C$ is found by the formula $a : b :: \sin. A : \underline{\sin. B}$ or $a : b :: \sin. A : \underline{\sin. AB_{,}C}$: hence the sines of the angles B and $AB_{,}C$ have the same numerical value (each being $=\dfrac{b \sin. A}{a}$) they are there-

fore the *supplements* (Note 1) of each other. This also appears from the figure; for since CB $=$ $CB_{,}$ the angle B and $CB_{,}B$ are equal: and the angle B is the supplement of $AB_{,}C$. Having then found the acute angle B subtract it from 180° for the angle $AB_{,}C$ in the other triangle: the remaining angles ACB $ACB_{,}$ and sides AB $AB_{,}$ may then be found by the rule.

EXAMPLE.

In the triangle ABC, given $a = 232$, $b = 345$, and A $=$ 37° 20′, to find the other parts.

Calculation of angle B.

$a : b :: \sin. A : \sin. B$

log. b. ..	2.537819
log. sin. A..	9.782796
	12.320615
log. a ..	2.365488
log. sin. B..	9.955127

First solution $B = 64°\ 24'$

Calculation of angle C.

$C = 180 - A - B$

$A = 37°\ 20'$
$B = 64\ \ 24$
$\overline{101\ \ 44}$
180
$\therefore C = 78\ \ 16$

Calculation of side c.

$c : a :: \sin. C : \sin. A$

log a ..	2.365488
log. sin. C..	9.990829
	12.356317
log. sin. A..	9.782796
log. c ..	2.573521

$\therefore c = 374.6$

There are two solutions,
In the first, B $= 64°\ 24'$ (fig. 3′)
In the sec., B $= 115\ 36$ (fig. 3″)

Second solu. $B = 115°\ 36'$

Calculation of angle C

$C = 180° - A - B$

$A = 37°\ 20'$
$B = 115\ \ 36$
$\overline{152\ \ 56}$
180
$\therefore C = 27\ \ 4$

Calculation of side c.

$c : a :: \sin. C : \sin A$.

log. a ..	2.365488
log. sin. C..	9.658037
	12.023525
log. sin. A..	9.782796
log. c ..	2.240729

$\therefore c = 174.07$

Examples.

Find the other parts of the triangle ABC having given:

(149)	$a=178.3$	Ans. $A=54^\circ\ 2'$ or $124^\circ\ 58'$
	$b=145$	$C=84\ \ 48$ or $12\ \ 52$
	$B=41^\circ\ 10'$	$c=219.32$ or 49.05
(150)	$a=2597.84$	$B=80^\circ\ 39'\ 45''$ or $99^\circ\ 20'\ 15$
	$b=3084.33$	$C=43\ \ 7\ \ 30$ or $24\ \ 27$
	$A=56^\circ\ 12'\ 45''$	$c=2136.7$ or 1293.7

IV.

Two sides and the included angle being given, to find the remaining angles.

To the log. tangent of half the supplement of the given angle, add the log. of the difference of the given sides: and from the sum, subtract the log. of the given sides: the result will be the log. tangent of half the difference of the required angles: which take from the tables.

To half the sum of the required angles, already known, *add* half their difference just found from the tables, and the sum will be the *greater* of the two angles required. To find the *less* of the two required angles, *subtract* the half difference.

Note. This case is more easily worked by expressing the above rule analytically: thus if a, b and C be the given quantities, then $a+b : a-b :: \tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B)$. The first three terms in this proportion

are known, (for $A+B=180-C$): hence the fourth term is easily found, which gives us half the difference of the required angles.

Example.

Given $a=798$ $b=460$ and $C=55° \, 2' \, 15''$: required the angles A and B. (see fig. 4)

$$a+b : a-b :: \text{tan. } \tfrac{1}{2}(A+B) : \text{tan. } \tfrac{1}{2}(A+B)$$

a .. 798	log. tan. $\frac{1}{2}(A+B)$ .. 10.283138
b .. 460	log. $(a-b)$ .. 2.528917
$a+b$ .. 1258	12.812055
$a-b$ .. 338	log. $(a+b)$ .. 3.099681
180° 0′ 0″	log. tan. $\frac{1}{2}(A-B)$ .. 9.712374
55 2 15 .. C	$\therefore \frac{1}{2}A-\frac{1}{2}B$ .. $=27° \, 16' \, 45''$
	and $\frac{1}{2}A+\frac{1}{2}B$ .. $=62 \;\; 28 \;\; 52$
2)124 57 45 .. $A+B$	
	$\therefore A$ .. $=89 \;\; 45 \;\; 37$
62 28 52 .. $\frac{1}{2}(A+B)$	and B .. $=35 \;\; 12 \;\; 7$

Examples.

Find the other two angles of the triangle ABC having given

(151) $a=798$ — Ans. $A=89° \, 45' \, 37''$
$b=460$ — $B=35 \;\; 12 \;\; 7$
$C=55° \, 2' \, 15''$

(152) $b=64$ — Ans. $B=56 \;\; 52 \;\; 15$
$c=70$ — $C=56 \;\; 47 \;\; 15$
$A=66° \, 20' \, 30''$

(153) $a = 512$ — Ans. $A = 54°\ 8'\ 11''$
$b = 627$ — $B = 82\ 58\ 11$
$C = 42°\ 53'\ 38''$

V.

Given two sides, and the included angle to find the third side.

Add together 10.602060, log. haversine of given angle, and logarithm of each given side: take half the sum, from which subtract log. of difference of the given sides. Look in the tables for the remainder as a log. tangent, and take out the corresponding log. sine, which subtract from the half sum just used. The remainder will be the log. of the required side.

NOTE. When the half sum is first found, write it down again a little to the right hand for the convenience of placing under it the log. sine to be subtracted

EXAMPLE.

Given $a=20$, $b=30$, and $C=100°$: required c. (fig. 5)

	Constant log. . . .	10·602060		
	log. hav. C.	[illegible]		
a. . 20 . . .	log. a. .	1·301030		
b. . 30. . .	log. b .	1·477121		
b. . 10		2)23·148719		
		11·574359.		11·743589
	log. $a-b$. .	1·000000		
	log. tan. . .	10·574359	log. sin.	9·985104
			log c .	1·589255
			c. .	38·84

Examples.

Find the third side in the following examples.

(154) $a=798$ Ans. $c=654$
$b=460$
$C=55^\circ\ 2'\ 16''$

(155) $c=48$ Ans. $b=73{\cdot}98$
$a=96$
$B=49^\circ\ 0'$

(156) $a=512$ Ans. $c=430$
$b=627$
$C=42^\circ\ 53'\ 38''$

(157) $b=2$ Ans. $a=1$.
$c=2\frac{1}{2}$
$A=22^\circ\ 20'\ 0''$

SOLUTION OF RIGHT ANGLED TRIANGLES.

Right angled triangles are solved by means of the trigonometrical ratios or expressions for the sine, tangent, &c. in art. 5 p. 4.

RULE.

Select that expression which contains two known terms and also the term required, and when possible let the numerator of the term selected be the part required. Reduce it to tabular logarithms (32) and then find the value of the unknown term (see Ex. 122).

Examples.

(1) In a right angled triangle ABC; (fig. 6) given $B=90^\circ$, $a=42$, and $A=50^\circ\ 10'$: required the other parts.

To find c. (fig. 6′) — Calculation.

(By art. 5. d).. $\frac{c}{a} = \cot. A$

or $c = a \cot. A$

$\therefore \log. c = \log. a + \log. \cot. A - 10$

$\log. a$ .. 1.623249
$\log. \cot. A$ .. 9.921247

$\log. c$ .. 1.544496
$\therefore c = 35.03$

To find b. (fig. 6″)

(By art. 5. b.) $\frac{b}{a} = \text{cosec}. A$

or $b = a \text{ cosec}. A$

$\therefore \log. b = \log. a + \log. \text{cosec} A - 10$

$\log. a$.... 1·623249
$\log \text{cosec}. A$.. 10.114689

$\log. b$.... 1·737938
$\therefore b = 54·7$.

To find C.

$C = 90 - A = 39° \, 50'$

Ex. 2. Given $a = ·02$ $b = ·1$ and $C = 90°$: required the other parts.

To find A. (fig. 7′) — *Calculation.*

(By art. 5. c.) $\frac{a}{b} = \tan. A$

$\log. a - \log. b = \log. \tan. A - 10$

$\therefore \log. \tan. A = \log. a + 10 - \log. b$

$\log. a$ .. $\bar{2}.301030$
10

8.301030
$\log. b$ .. $\bar{1}.000000$

$\log. \tan. A$.. 9.301030
$\therefore A = 11° \, 18' \, 30''$

To find c. (fig. 7″) *Calculation.*

(By art. 5. e) $\frac{c}{b} = \sec. A$ log. b.... $\bar{1}\cdot000000$

or $c = b \sec. A$ log. sec. A.. 10.008514

$\therefore$ log. $c = \log. b + \log. \sec. A - 10$

9.008514

10

log. c.... $\bar{1}\cdot008514$

$\therefore c = .102$

Examples.

Find the other parts of the right angled triangle ABC, having given

(158) $A = 52° 38' 0''$ Ans... $C = 37° 22'$
$b = 45$ $a = 35.76$
$B = 90°$ $c = 27.31$

(159) $A = 49° 14'$ Ans... $C = 40° 46'$
$c = 331$ $a = 384$
$B = 90°$ $b = 506.8$

(160) $A = 56° 29' 15''$ Ans... $a = 3555$
$b = 4264.3$ $c = 2354$
$B = 90°$ $C = 33° 30' 45''$

(161) $A = 4° 44'$ Ans... $c = 8390$
$a = 694.73$ $b = 8419$
$B = 90°$ $C = 85° 16'$

(162) $b = \cdot 2$ Ans. . $a = \cdot 1286$
$A = 40°$ $c = \cdot 1532$
$B = 90°$ $C = 50°$

(163)	$c = \cdot 04$	Ans. $a = \cdot 04767$
	$C = 40^\circ$	$b = \cdot 06223$
	$B = 90^\circ$	$A = 50^\circ$
(164)	$a = 1777.5$	Ans. $A = 56^\circ\ 29'\ 15''$
	$c = 1277$	$C = 33\ \ 80\ 45$
	$B = 90^\circ$	$b = 2132.1$

VI.

Two sides and the included angle being given to find the area.

Add together logarithms of the two given sides, and log. sine of the given angle; the sum, rejecting 10 in the index will be the logarithm of twice the required area.

Examples.

Given $a = 798$, $b = 460$ and $C = 55^\circ\ 2'\ 15''$ required the area.

log. a	2·902003
log. b	2·662758
log. sin. C	9·913563
log. 2 area.....	5·478324
∴. 2 area.....	300832
and area.......	150416

(165) Given $a = 245$ yards, $b = 760$ yards and $C = 60^\circ$ required the area. Ans. 80627 sq. yards.

VII.

Three sides of a plane triangle being given, to find the area.

From half the sum of the three sides, subtract each side separately. Add together, the log. of the half sum, and the logarithms of the three remainders. Half the result will be the logarithm of the area.

Example.

166. Given $a=798$, $b=460$ and $c=654$: required the area.

a..798				log. 956..2.980458
b..460	956	956	956	log. 158..2.198657
c..654	798	460	654	log. 496..2.695482
——	——	——	——	log. 302..2.480007
2)1912	log. 158	log. 496	log. 302	——
——				2)10.354604
log. 956				——
			Nat. No. 150418	5.177302

(167). In the trapezium $ABCD$, fig. (8) $AB=90$ yds, $BC=100$ yards, $CD=110$ yards, $DA=120$ yards, the diagonal $BD=178.8$ yards: required the area of the trapezium. Ans. 9768.7 square yards.

RULES IN SPHERICAL TRIGONOMETRY.

VIII.

Three sides of a spherical* triangle being given, to find an angle. (using haversines).

Put down the two sides containing the required angle, and take the difference, under which put the third side: take the sum and difference.

Add together the log. cosecants of the two first terms in this form (rejecting the tens from the index,) and the *halves* of the log. haversines of the two last terms. The result will be the log. haversine of the required angle.

IX.

(Without using haversines).

Three sides of a spherical triangle being given, to find an angle.

Put down the two sides containing the required angle, and take the difference: under which put the third side: take the sum and difference, and the half sum and half difference.

Add together the log. cosecants of the two first terms in this form, (rejecting the tens from the

* A *spherical* triangle is that part of the surface of a sphere which is bounded by arcs of three *great* circles, or three circles whose planes pass through the center of the sphere. The three arcs are the *sides* of the triangle; and any of its *angles* is the same as the inclination of the planes of the sides containing the angle.

index,) and the log. sines of the two last terms; divide by 2, and look out the result as a log. sine: the angle corresponding to which will be half the required angle.

168. In the spherical triangle ABC, (fig. 9) given $a=124°\ 10'$, $b=89°\ 0'\ 15''$, and $c=108°\ 40'$, required the angle A.

BY RULE VIII.

	$89°\ 0'\ 15''$	log. cosec. . . . 0·000066
	108 40 0	log. cosec. . . . 0·023468
	19 39 45	$\frac{1}{2}$ log. hav. S. . 4·978000
	124 10 0	$\frac{1}{2}$ log. hav. D. . 4·898018
(S) . .	143 49 45	
(D) . .	104 30 15	log. hav. A. . 9·899552
		∴ A. . $=125°\ 56'\ 45''$

BY RULE IX.

	$89°\ 0'\ 15''$	log. cosec. . . . 0·000066
	108 40 0	log. cosec . . . 0·023468
	19 39 45	log. sin. ($\frac{1}{2}S$ 9·978000
	124 10 0	log, sin. ($\frac{1}{2}D$) 9·898018
S. .	143 49 45	2)19·899552
D. .	104 30 15	log. sin. $\frac{1}{2}A$. 9·919776
$\frac{1}{2}S$. .	71 54 52	∴ $\frac{1}{2}A=62°\ 58'\ 15''$
$\frac{1}{2}D$. .	52 15 7	2
		and A. . $=125\ 56\ 30$

Examples.

Find the three angles of the spherical triangle ABC, having given

(169)	$a=$ 49	10	0	Ans.	$A=$ 59	2	0
	$b=$ 58	25	0		$B=$ 74	54	0
	$c=$ 56	42	0		$C=$ 71	18	30
(170)	$a=$119	42	20	Ans.	$A=$115	38	45
	$b=$108	4	18		$B=$ 99	21	15
	$c=$ 68	53	42		$C=$ 75	31	30
(171)	$a=$ 87	10	15	Ans.	$A=$ 81	24	0
	$b=$ 62	36	45		$B=$ 61	31	15
	$c=$100	10	15		$C=$102	59	0

X.

Two sides, and the included angle being given, to find the remaining side. (using haversines)

Add together 6.301030, the log. sines of the given sides, and log. haversine of given angle: reject 30 from the index of the sum, and take out the natural number of the resulting logarithm. To this natural number add the natural versine of the difference of the given sides: the sum will be the natural versine of the required side; which find from the tables.

XI.

(Without using haversines).

Two sides and the included angle being given to find the remaining side.

Add together 6.301030, the log. sines of the given sides, and twice the log. sine of half the

given angle; rejecting 40 from the index of the sum, and take out the natural number of the resulting logarithm. To this natural number, add the natural versine of the difference of the given sides: the sum will be the natural versine of the required side, which find from the table.

Ex. In the spherical triangle *ABC* (fig. 9) given $c = 119° 42' 20''$, $b = 108° 4' 18''$, and $A = 75° 31' 30''$: required the remaining side a.

By Rule X.

	const. log. ..	6.301030
$c = 119° 42' 20''$	.. log. sin. c	9.938818
$b = 108 \quad 4 \quad 18$	... sin. b	9.978031
	.... hav. A....	9.574056
$c - b = 11 \quad 38 \quad 2$		
	log.......	5.791935
	Nat. No......	619349
	Nat. ver. $(c-b)$	20544
	Nat. ver. a	639893
	$\therefore a$ .. $= 68° 53' 36''$	

By Rule XI.

	const. log. ..	6.301030
$c = 119° 42' 20''$	.. log. sin. c .. .	9.938818
$b = 108 \quad 4 \quad 18$	.... sin. b ...	9.978031
	. sin. $\frac{1}{2} A$	9.787028
$c - b = 11 \quad 38 \quad 2$	. sin. $\frac{1}{2} A$ ..	9.787028
$\frac{1}{2} A = 37° 45' 45''$	.. log..... .	5.791935
	Nat. No..... .	619349
	Nat. ver. $(c-b)$	20544
	Nat. ver. a	639893
	$\therefore a$ .. $= 68° 53' 36''$	

EXAMPLES.

Find the third side of the spherical triangle *ABC*, having given.

(172)	$A = 96° \; 32' \; 0''$		Ans.	$a = 96° \; 9' \; 57''$	
	$b = 76 \; 42 \; 0$				
	$c = 89 \; 10 \; 30$				
(173)	$A = 50 \; 0 \; 0$		Ans.	$a = 46 \; 19 \; 32$	
	$b = 70 \; 45 \; 10$				
	$c = 62 \; 10 \; 15$				
(174)	$a = 100 \; 8 \; 42$		Ans.	$c = 87 \; 0 \; 50$	
	$b = 98 \; 10 \; 5$				
	$C = 88 \; 24 \; 24$				
(175)	$b = 118 \; 2 \; 14$		Ans.	$a = 23 \; 57 \; 9$	
	$c = 120 \; 18 \; 33$				
	$A = 27 \; 22 \; 34$				
(176)	$a = 87 \; 10 \; 15$		Ans.	$c = 100 \; 9 \; 38$	
	$b = 62 \; 36 \; 45$				
	$C = 102 \; 58 \; 30$				
(177)	$a = 69 \; 19 \; 10$		Ans.	$c = 104 \; 59 \; 57$	
	$b = 78 \; 59 \; 14$				
	$C = 110 \; 48 \; 42$				

XII.

Of two sides and the opposite angles, any three being given to find the fourth.

Write down a proportion having for the two first terms the sines of the two sides concerned, and for the third term the sine of the angle opposite to the first side put down, and for the fourth term the sine of the angle opposite to the other

side put down in the proportion. Mark the term required, and proceed as in the corresponding rule for plane triangles.

EXAMPLE.*

(1) In the spherical triangle *ABC*, given $a = 70° 10' 30''$ $b = 80° 5'$, and $B = 33° 15'$: required the angle *A*.

(2) $b = 119° 42' 20''$, $a = 108° 4' 18''$, and $A = 99° 21' 30''$: required *B*.

sin. *a* : sin. *b* :: sin. *A* : sin. *B*

log. sin. *a* ..	9.973466
sin. *B* ..	9.739013
	19.712479
sin. *b*	9 993462
sin. *A* ..	9.719017
∴ *A* ..	= 31° 34′ 30″

In this example *a* being less than *b*, *A* is less than *B*, and therefore *A* is less than 90°.

sin. *a* : sin. *b* :: sin. *A* : sin. *B*

log. sin. *b* ..	9.938818
sin. *A* ..	9.994181
	19.932999
sin. *a*..	9.978031
sin. *B*..	9.954968
. *B* ..	= 180° − 64° 21′ 30″
	= 115° 38′ 20″

In this example *b* being greater than *a*, *B* is greater than *A*, and therefore *B* is greater than 90°: hence the angle taken out of the tables must be subtracted from 180° to get *B*.

* Unless by some limitation of this kind it should be determined that the required angle is less or greater than 90°, this rule is considered ambiguous. (See corresponding rule for plane triangles.

XIII.

Two sides and the included angle being given to find the other two angles.

Take the sum and difference of the two given sides, and the half sum and half difference, and also half the given angle.

Under heads (1) and (2) put down the following logarithms.

Under (1) and (2) log. cot. of half the given angle.

Under (1) log. cosec. } of half the sum of the given sides.
Under (2) log. sec. }

Under (1) log. sin. } of half the difference of given sides.
Under (2) log. cos. }

Add together the logarithms under (1) and (2), and (rejecting 20 from each index) the results will be the log. tangents of half the difference and half the sum (*a*) of the required angles respectively.

The sum of the two arcs thus found will be the angle opposite to the greater of the given sides: and their difference will be the angle opposite to the less of the given sides.

(*a*) Note. If half the sum of the given sides be greater than 90°, half the sum of the required angles will also be greater than 90°: in this case, the arc found as above under (2) must be subtracted from 180° to get half the sum of the required angles.

EXAMPLE.

(178). In the spherical triangle ABC given $a=124°\ 10'$, $b=89°\ 0'\ 15''$, $C=112°\ 1'\ 30''$: required the other two angles A and B.

$a=124°\ 10'\ 0''$		
$b=\ \ 89\ \ \ 0\ \ 15$		
$a+b$ .. 213 10 15	(1)	(2)
$a-b$ .. 35 9 45	cot. $\frac{1}{2}C$.... 9.828783	cot. $\frac{1}{2}C$ .. 9.828783
$\frac{1}{2}(a+b)$.. 106 35 7	cosec $\frac{1}{2}(a+b)$ 0.018455	sec $\frac{1}{2}(a+b)$ 0.544480
$\frac{1}{2}(a-b)$.. 17 34 52	sin. $\frac{1}{2}(a-b)$ 9.480090	cos. $\frac{1}{2}(a-b)$ 9.979225
	tan. $\frac{1}{2}(A-B)$ 9.327328	tan. (arc.) 10.352488
	tan. $\frac{1}{2}(A-B)$ .. 11° 59′ 45″	arc. 66° 3 0″
		180
		$\frac{1}{2}(A+B)$ .. 113 57 0
		$\frac{1}{2}(A-B)$ .. 11 59 45
		A .. 125 56 45
		B .. 101 57 15

(179) Given $b=89°\ 0'\ 15''$, $c=108°\ 40'$, and $A=125°\ 56'\ 45''$ to find the angles B and C.

Ans. $B=101°\ 57'\ 15''$
$C=112\ \ 2\ \ 0$

Examples to this and the following rule are easily formed out of those already given. (See Ex. 169, &c.)

XIV.

Given two angles and the included side, to find the other two sides.

Take the sum and difference of the two given angles: and the half sum and half difference: and also half the given side.

Under heads (1) and (2) put down the following logarithms.

Under (1) and (2) log. tan. of half the given side.
Under (1) log. cosec. } of half the sum of given angles.
Under (2) log. sec.
Under (1) log. sin. } of half the difference of given angles.
Under (2) log. cos.

Add together logarithms under (1) and (2), and (rejecting 20 in each index) the results will be log. tangents of ½ the difference and ½ the sum (*a*) of required sides respectively. The sum of the two arcs will be the side opposite the greater of the given angles: and their difference will be the side opposite to the less of the given angles.

(a) NOTE. If half the sum of the given angles exceed 90°, the arc taken out under (2) must be subtracted from 180°, to give half the sum of the required sides.

EXAMPLE.

(180) In the spherical triangle ABC given $A=113°\ 33'\ 30''$, $B=51°\ 30'\ 20''$, and $c=60°\ 18'\ 4''$: required the other two sides.

$A=$	113	33	30				
$B=$	51	30	20				
$A+B$	165	3	50	(1)		(2)	
$A-B$	62	3	10	tan. $\frac{1}{2}c$	9.764070	tan. $\frac{1}{2}c$	9.764070
$\frac{1}{2}(A+B)$	82	31	55	cosec. $\frac{1}{2}(A+B)$	0.003709	sec. $\frac{1}{2}(A+B)$	0.886220
$\frac{1}{2}(A-B)$	31	1	35	sin. $\frac{1}{2}(A-B)$	9.711935	cos. $\frac{1}{2}(A-B)$	9.932952
$\frac{1}{2}c$	30	9	2	tan. $\frac{1}{2}(a-b)$	9.479725	tan. $\frac{1}{2}(a+b)$	10.583242
				$\frac{1}{2}(a-b)$ ..	16° 47′ 45″	$\frac{1}{2}(a+b)$ ..	75° 22′ 15″
						$\frac{1}{2}(a-b)$ ..	16 47 45
						a....	92 10 0
						b....	58 34 30

XV.

Right angled spherical triangles.

In a right angle spherical triangle *ABC* (fig. 10), *A* being the right angle, the two sides *b* and *c*, and the *complements** of the three other parts, namely, *co. a*, *co. B* and *co. C* are called the *five circular parts.* By the following rules, called from the name of their author, *Napier's Rules*, any two of these parts being given, the other three may be found.

In applying the rules, we must select one of the three parts concerned, such, that the other two may either be *both adjacent* to it, or *both separate* from it. The part so selected is called the *middle part.* Rule I. will apply to the former case; Rule II. to the latter.

* The *complement* of an angle is what it wants of 90°: thus in fig. Art. 5, the angle P is the complement of the angle A. It may be easily proved that the sin. tan. sec. cos. cot. and cosec. of an angle are the cos. cot. cosec. sin. tan. and sec. of its complement respectively. Thus let the two right angled triangles CPN $C_{/}P_{/}N_{/}$ (fig. 11,) be equal to each other in every respect: that is $C = C_{/}$, $P = P_{/}$, and $N = N_{/}$, then $P_{/}$ is the complement of A or co. A.

$$\text{Now (Art. 5) sin. A} = \frac{PN}{CP} = \frac{P_{/}N_{/}}{C_{/}P_{/}} = \text{cos. } P_{/} = \text{cos. co. A.}$$

$$\text{tan. A} = \frac{PN}{CN} = \frac{P_{/}N_{/}}{C_{/}N_{/}} = \text{cot. } P_{/} = \text{cot. co. A \&c.}$$

Hence in applying the above rules, sin. A is substituted for cos. co. A; tan. A for cot. co. A, &c.

Rule I.

The *sine* of the middle part is equal to the product of the *tangents* of the two parts *adjacent* to it.

Rule II.

The sine of the middle part is equal to the product of the *cosines* of the two parts *opposite to,* or *separated* from it.

Having written down the equation according to the case, make a dash under the part required, and determine its magnitude by applying the proper signs (+ or —) to each term according to the rule in Art. 34.

Examples.

(1). In the right angled spherical triangle ABC (fig. 12) given $b=74^\circ\ 19'\ 30''$, $c=38^\circ\ 56'$, and $A=90^\circ$: required the other parts.

To find B (fig. 12′), the right angle A not being considered as a part, c will be the middle part, and complement B and b are the *adjacent* parts.

$\therefore$ By Rule I.

sin. c=tan. co. B tan. b	*Calculation.*
or sin. c=$\underline{\text{cot. } B}$ tan. b	log. sin. c+10 . . 19.798247
(since tan. $\overline{\text{co. } B}$=cot. B	log. tan. b. 10.551887
see note page 61) and determining the	————
sign of B by art. 34 we have	log. cot. B . . 9 246360
$\overset{+}{\text{sin. } c}=\overset{+}{\text{cot. } B}\ \overset{+}{\text{tan. } b}$	$\therefore B$ . . $=80^\circ$

To find C (fig. 12″); marking the figure in the usual manner we see that b is the middle part, and c and C the adjacent parts.

∴ by Rule I.	*Calculation*
$\sin. b = \tan c \tan \text{co.} C$	log. sin. $b+10$ .. 19.983540
$+ \quad + \quad +$	.. tan. c .. 9.907336
or $\sin. b = \tan c \cot. \underline{C}$	
	.. cot. C .. 10.076204
	∴ C.. $=40°$

To find a fig. (12″). In this case a is the middle part and b and c are separated from it.

∴ by Rule II.	*Calculation.*
$\sin. \text{co.} a = \cos. b \cos. c$	log. cos. b.. 9.431654
$+ \quad + \quad +$	.. cos. c.. 9.890911
or $\underline{\cos. a} = \cos. b \cos. c$	
	log. cos. a (rejecting 10) 9.322565
	∴ a .. $= 77° \ 32'$

(2). Given $A=90°$, $B=78° \ 10'$, and $C=131° \ 32' \ 45''$, to find the other parts.

To find a fig. (13′)	To find b fig. (13″)	To find c fig. (13‴)
$\overset{-}{\underline{\cos. a}} = \overset{+}{\cot. B} \overset{+}{\cot. C}$	$\overset{+}{\cos. B} = \overset{+}{\underline{\cos. b}} \overset{+}{\sin. C}$	$\overset{-}{\cos. C} = \overset{+}{\sin. B} \overset{-}{\underline{\cos. c}}$
log.cot. B..9.321222	cos. $B+$ 10..19.311893	cos. $C+$10..19.821657
cot. C..9.947508	sin. C........ 9.874148	sin. B......9.990671
cos. a..9.268730	cos. b........ 9.437745	cos. c......9.830986
79° 18′ 0″	∴ b.... $=$ 74° 5′ 45″	47 20 30
180		180
∴ a.. $=$100 42		c.. $=$132 39 30

EXAMPLES.

In a right angled spherical triangle ABC, (fig. 10) find the other parts, having given

(181)	$B=72°\ 19'$	Ans. $a=54°\ 28'\ 0''$
	$b=50\ 50$	$c=23\ 2'\ 15$
	$A=90$	$C=28\ 45\ 0$
(182)	$b=60°\ 10'$	$B=60°\ 32'\ 45''$
	$c=100$	$C=98\ 41\ 45$
	$A=90$	$a=94\ 57\ 15$
(183)	$B=100°\ 0'$	$a=90°\ 30'\ 0''$
	$C=\ 87\ 10$	$b=100\ 23\ 15$
	$A=\ 90$	$c=87\ 7\ 15$
(184)	$c=\ 46°\ 18'\ 23''$	$b=26°\ 23'\ 15''$
	$B=\ 34\ 27\ 30$	$a=51\ 46\ 15$
	$A=\ 90$	$C=66\ 59\ 30$
(185)	$c=118°\ 21'\ 4''$	$b=116°\ 18\ 0''$
	$A=\ 23\ 40\ 12$	$C=100\ 59\ 30$
	$B=\ 90$	$a=\ 21\ 5\ 45$
(186)	$a=100°\ 42'$	$C=131°\ 32'\ 45''$
	$B=\ 78\ 10$	$c=132\ 39\ 30$
	$A=\ 90\ 0$	$b=\ 74\ 5\ 45$
(187)	$c=\ 53°\ 14'\ 20''$	$b=\ 91°\ 4'\ 15''$
	$A=\ 91\ 25\ 58$	$C=\ 53\ 15$
	$B=\ 90$	$a=\ 91\ 47\ 15$

XVI.

Quadrantal Spherical Triangles.

A spherical triangle having one *side* 90° is called a *Quadrantal* triangle.

The five circular parts in quadrantal triangles

are the two angles adjacent to the quadrant, and the complements of the other three: thus, if ABC (fig. 14), be a quadrantal triangle, the side a being 90°, the five circular parts are the angles B and C, and the complements of the third angle A and of the sides b and c.

Any two of these parts being given a third may be found by the preceding rules for right angled triangles (p. 62).

The magnitude of the part required is determined by the same rule as that given in Art. (34), observing that *when two angles or two sides come together on the same side of the equation*, the sign — must be placed before them, and the three signs thus placed on one side of the equation must be made to produce the same result (positive or negative) as the sign of the other side.

EXAMPLE.

In the quadrantal triangle ABC (fig. 15), given $B = 80^\circ\ 10'$, $C = 48^\circ\ 50'$ and $a = 90^\circ$; find the other parts.

To find A (fig. 15′)	To find b (fig. 15″)	To find c (fig. 15‴)
$\overset{-}{\underline{\cos. A}} = -\overset{+}{\cos. B}\ \overset{+}{\cos. C}$	$\overset{+}{\sin. C} = \overset{+}{\tan. B}\ \overset{+}{\underline{\cot. b}}$	$\overset{+}{\sin. B} = \overset{+}{\tan. C}\ \overset{+}{\underline{\cot. c}}$
cos. B.. 9.232444	sin. C+10.. 19·876678	sin. B+10.. 19.993572
cos C.. 9.818392	tan. B.. 10.761128	tan. C.. 10.058287
cos. A.. 9.050836	cot. b .. 9.115550	cot. c.. 9.935285
$83^\circ\ 32''\ 45''$	b.. $82^\circ\ 34'\ 0''$	c.. $49^\circ\ 15'\ 51''$
180		
A.. 96 27 15		

EXAMPLES.

In the quadrantal spherical triangle ABC (fig. 14), find the other parts; having given

(188)	$a=90°$ $A=100$ $c=50\ 10'$	Ans. $B=74°\ 36'\ 30''$ $b=78\ 14\ 30$ $C=49\ 8\ 0$
(189)	$B=45°$ $c=72$ $a=90$	Ans. $A=107°\ 10'\ 15''$ $b=47\ 44\ 30$ $C=65\ 19\ 15$
(190)	$b=90°$ $a=100$ $c=82\ 10\ 30$	Ans. $B=88°\ 36'\ 45''$ $A=100\ 5\ 45$ $C=82\ 3\ 15$
(191)	$a=90°$ $B=80\ 10$ $C=50\ 2$	Ans. $A=96°\ 17'\ 45''$ $b=82\ 26\ 0$ $c=50\ 27\ 0$
(192)	$A=72°\ 49'\ 45''$ $b=47\ 44\ 30$ $a=90$	Ans. $C=65°\ 19'\ 15''$ $B=44\ 59\ 45$ $c=72\ 0\ 0$
(193)	$c=49°\ 23'\ 45''$ $b=76\ 41$ $a=90$	Ans. $A=101°\ 42'\ 15''$ $B=72\ 20\ 30$ $C=48\ 1\ 50$
(194)	$a=60°\ 10'\ 15''$ $b=80\ 20\ 30$ $c=90$	Ans. $A=59°\ 41'\ 45''$ $B=78\ 51\ 0$ $C=95\ 36\ 0$

End of Rules.

Examination Papers in preceding Rules.

(1)

(195) Required the product of $18 \times 48 \times 6.2 \times 4$

Ans. 21427

(196) Divide 236 by 16.1 . . . . 14.66

(197) Required the 3rd power of 12.5 . . . . 1953.125

(198) . . . . 4th . . . . ·076543 . . . . ·000034326

(199) . . . . 6th root of 11078 . . . . . . 4.72146

(200) . . . . 7th . . . . ·098674 . . . . ·7183146

(201) . . . . value of $(19)^{\frac{2}{5}}$ 3.247

(202) . . . . value of x in the following proportion;

$24 : 17.5 :: 79 : x$ . . . . . . $x = 57.604$

(203) Reduce the following expression to logarithms.

$x = \frac{a^2 b}{cd}$ . . Ans. $\log. x = 2 \log. a + \log. b - \log. c - \log. d$.

(204) Find the value of x in the following equation,

$2^x = 769$ . . . . . . Ans. $x = 9.5868$.

(205) The first term of a geometrical series is 2, its common ratio 3, and number of terms 8: find the sum.

Ans. $S = 6560$.

(206) If £400 be placed out at compound interest for nine years, at £4 per cent., per annum: required its amount. Ans. £569 6 8.

(2)

(207) In the plane triangle ABC given $a = 10$, $b = 15$, and $c = 20$: required the angle A. Ans. $A = 28° 57' 15''$.

(208) In the plane triangle ABC given $a = 25{\cdot}125$ $b = 12\frac{1}{2}$, and $A = 68° 48'$: to find B.

Ans. $B = 27° 38' 15''$.

(209) Given $a = 399$, $b = 230$, and $C = 55° 2' 15''$: required the angle A. Ans. $A = 89° 45' 37''$.

(210) Given $a = 40$, $b = 60$, and $C = 100°$: required c. Ans. $c = 77.68$.

(211) In the right angled triangle ABC, given $B = 90°$ $a = 210$, and $A = 50° 10'$: required the other parts.

Ans. $C = 39° 50'$, $c = 175.1$, $b = 273.5$.

(212) Given $a = 399$, $b = 230$, and $C = 124° 57' 45''$, required the area. Ans. 37604.

(213) Given $a = 399$, $b = 230$, and $c = 327$: required the area. Ans. 37604·7.

(214) Give definitions of the sine, cosine, tangent, cotangent, secant, cosecant, and versine of an angle.

(3)

(215) In the spherical triangle ABC, given $a = 120° 54'$, $b = 105° 6' 0''$, and $c = 108° 41' 30''$: required the angles A, B, and C.

Ans. $A = 130° 50'$, $B = 121° 35'$ and $C = 123° 18'$.

(216) In the spherical triangle ABC, given $a = 64° 21' 15''$, $b = 80° 38' 45''$, and $c = 104° 28' 30$: required the angles A, B, and C.

Ans. $A = 60° 17' 45''$, $B = 71° 55' 45''$, $C = 111° 6' 15''$.

(217) Given $a = 87° 10' 15''$, $c = 100° 10' 15''$, and $B = 61° 31' 15''$: required b. Ans. $b = 62° 36' 45''$.

(218) Given $a = 81° 10' 0''$, $A = 81° 24'$, and $b = 62° 36' 45''$: required B. Ans. $B = 62° 41' 0''$.

(219) Given $a = 49° 10'$, $b = 58° 25'$, and $C = 71° 18' 30''$: required the angles A and B.

Ans. $A = 59° 2'$, $B = 74° 54'$.

(220) Given $a = 87° 10' 15''$, $c = 100° 10' 15''$, and $B = 61° 31' 15''$: required the angles A and C.

Ans. $A = 81° 24'$, $C = 102° 59'$.

(221) Given $A = 59° 2'$, $B = 74° 54'$, and $b = 56° 42'$: required the sides a and b.

Ans. $a = 49° 10'$, $b = 58° 25'$.

(222) Given $A = 115° 38' 45''$, $C = 75° 31' 30''$, and $b = 108° 4' 15''$: required the sides a and c.

Ans. $a = 119° 42' 15''$, $c = 68° 53' 45''$.

(4)

(223) In the right angled spherical triangle ABC, given $B = 60° 32' 15''$, $a = 94° 57' 20''$, and $A = 90°$: required the other parts.

Ans. $C = 98° 41' 45''$, $c = 100° 0'$, $b = 60° 10'$.

(224) Given $a = 77° 52' 10''$, $b = 74° 19' 30''$, and $A = 90°$, to find the other parts.

Ans. $c = 38° 56' 24''$, $B = 80°$, $C = 40°$.

(225) In the quadrantal triangle ABC, given $a = 90°$, $B = 74° 36' 30''$, and $c = 50° 10'$: required the other parts.

Ans. $A = 100° 0'$, $b = 78° 14' 25''$, $C = 49° 8\ 15''$.

(226) Given $a = 90°$, $A = 96° 17' 52''$, $b = 82° 26'$: required the other parts.

Ans. $B = 80° 10'$, $C = 50° 2'$, $c = 50° 27'$.

(5)

(227) Required the product of $2{\cdot}4 \times {\cdot}0035 \times 1.08 \times {\cdot}1$

Ans. ·0009072

(228) Divide 9·5 by ·36. Ans. 26·389.

(229) Find the value of $\dfrac{8{\cdot}4 \times {\cdot}0769 \times {\cdot}00683}{59{\cdot}8 \times {\cdot}0000146 \times {\cdot}0039}$

Ans. 1295·71.

(230) Find the third power of ·2321 and the 72nd power of ·96797. Ans. ·0125 and ·096.

(231) Required the 200th root of ·0063241.

Ans. ·975.

(232) Required the value of $\sqrt[{\cdot}27]{{\cdot}6958825}$

Ans. ·00563.

(233) Required the value of the following expression,

$\sqrt{\dfrac{\sqrt[{\cdot}72]{{\cdot}096} + ({\cdot}096)^{{\cdot}72}}{2{\cdot}5}}$ Ans. ·29904.

(234) In the plane triangle *ABC*, given $a = \frac{1}{8}$, $b =$ ·2705, and $c =$ ·3375 : required the angle *C*.

Ans. $C = 111^\circ\ 29'\ 23''$.

(235) In the plane triangle *ABC*, given $a = 116$, $b = 172.5$, and $A = 37^\circ\ 20'$, to find *B*.

Ans. $B = 64^\circ\ 24'$ or $115^\circ\ 36'$.

(236) In the plane triangle *ABC*, given $a =$ ·512 $b =$ ·627, and $C = 42^\circ\ 53'\ 38''$: required the angles *A* and *B*. Ans. $A = 54^\circ\ 8'\ 11''$. $B = 82^\circ\ 58'\ 11''$.

(237) Given $b =$ ·2, $c =$ ·25, and $A = 22^\circ\ 20'$: required the third side *a*. Ans. $a =$ ·1.

(238) In the right angled plane triangle *ABC*, given $a = 177\frac{3}{4}$, $c = 127.7$, and $B = 90^\circ$: required the side *b*.

Ans. $b = 21321$.

(239) In the trapezium *ABCD*, fig. (8) given the side $AB = 90$ yards, $BC = 100$ yards, $CD = 110$ yards, $DA = 120$ yards, and the angle $DAB = 116^\circ\ 0'$: required the area. Ans. Area = 9768 7 yards.

PROBLEMS

IN PLANE AND SPHERICAL TRIGONOMETRY.

When a line joining any two points in space is accessible throughout its whole extent, it may in general be measured by the successive application of some line of a known length: but when it is inaccessible, or cannot be directly measured, we may obtain its length by considering it the *side of a triangle*, if we already know, or can find by observation, a sufficient number of parts of the triangle to enable us to apply the preceding rules. It is thus that the mensuration of inaccessible lines of any length is found by means of that of accessible lines and angles.

The angle DOC, (fig. 16) contained by lines drawn from a point O to two remote objects C and D may be measured by placing a circle in the plane passing through the two objects, and having its center at the angular point O. The straight edge of a ruler being then placed on the circle so as to pass through its center, and by means of sights placed over it, directed first to one object and then to the other; the arc AB of the circumference between the two positions can be observed: this is manifestly the measure of the angle O.

The principal instruments for measuring angles are the theodolite and sextant. A theodolite is the most convenient instrument for measuring horizontal and vertical angles: it is composed of two circles having their planes perpendicular to each other. When the instrument is used, one of the circles is placed in a horizontal plane, by means of levels; on this circle horizontal angles are measured:—on the other are measured vertical angles, whether of elevation or depression, (that is, whether one of the objects is above or below the horizon). A sextant is employed to measure angles contained in any plane whatever. It is more suited for observing angular distances of heavenly bodies than the theodolite; but the latter is better adapted for surveying than the former, since it determines the horizontal angles at once; but those observed with the sextant must, when out of the plane of the horizon, be reduced to that plane by calculation, to suit them to the purposes of the survey.

We will not stop to give particular descriptions of the theodolite and sextant: their construction will be best learned by a careful study of the instruments themselves: we shall therefore suppose the manner of adjusting and applying them to practice is known, and proceed to give a collection of problems which they enable us to solve.

Problems in Plane Trigonometry.

(1) (fig. 17). On the opposite bank of a river to that on which I stood, is a tower known to be 216 feet high;

with a pocket sextant I ascertained the angle between a horizontal line drawn from my eye, (supposed to be 5 feet above the ground) and its top to be $47° 56'$: required the distance across the river, from the place where I stood, to the bottom of the tower. Ans 190·4 feet.

(2) (fig. 18). A may-pole being broken off by the wind, its top struck the ground at 15 feet distance from the foot of the pole: required the height of the whole may-pole, supposing the length of the broken piece to be 39 feet Ans. 75.

(3) (fig. 19). A ladder 36 feet long, may be so placed that it shall reach a window 30.7 feet from the ground on one side of the street, and by only turning it over, without moving the foot out of its place, it will reach another window 18.9 feet high on the other side. What is the breadth of the street, and the angle of elevation of the first window from the second.

Ans. Breadth of street 49·44
Angle of elevation $13° 25'$.

(4) (fig. 17). From the bottom of a tower, a distance $AB = a$ yards is measured on a horizontal plane, and at A, the angle BAC is found $= d°$: required the height of the tower BC.

First, when $a = 50$ and $d° = 25° 17'$. Ans. 23.62
Second, .. $a = 12\frac{1}{4}$ and $d° = 45°$. Ans. $12\frac{1}{4}$

(5) (fig. 20). To determine the distance of a ship at anchor at C, I measured a straight line $AB = a$ yards on the shore, and observed the angles $CAB = d°$ and $CBA = d_{,}°$: required the distance of the ship from A.

First, when $a = 1000$ yds. $d^\circ = 32^\circ\ 10'$ and $d_,^\circ = 83^\circ\ 18'$.
Ans. 1104.7.

Second, .. $a = 2{\cdot}5$ miles $d^\circ = 80^\circ$ and $d_,^\circ = 50^\circ$.
Ans. 2.5 miles.

(6) (fig. 20). Two ships sailing in company in order to determine nearly their distance from an object C on the shore, are separated from each other two nautical miles AB: the angle is then observed from each ship between the object and the other ship: at A it is $85^\circ\ 10'$ at B $82^\circ\ 45'$: required the distance of each ship from the object. Ans. 9.478 and 9.52.

(7) (fig. 21). To determine the height AB of a tower inaccessible at the base, two stations C and D are chosen in a horizontal plane, so that a plane passing through BC and D is perpendicular to the horizon; the distance $CD = a$ yards, and the angles $ACB = d$, and $BDA = d_,$: required the height AB.

First, when $a = 100$ yards, $d^\circ = 46^\circ\ 15'$, $d^\circ_, = 31^\circ\ 23'$
Ans. 145.9

Second, when $a = 300$ yards, $d^\circ = 58^\circ$, and $d^\circ_, = 32^\circ$
Ans. 307.4

(8) (fig. 21). From the decks of two ships at D and C, 880 yards asunder, the angle of elevation of a cloud at A which bears on the same point of the compass from each is observed, at D the angle is 35°, at C it is 64°: required the height of the cloud above the surface of the sea, the deck of each ship being supposed to be elevated above it 21 feet. Ans. 942.6 yards.

(9) (fig. 20). To determine nearly the distance between two ships at sea, I carefully observed the interval

of time between the flash and report of a gun from each, and measured the angle which the two ships subtended. The intervals were a seconds and b seconds: and the angle observed $= d^\circ$: required the distance of the ships.

First, when $a = 4$ sec. $b = 6$ sec. and $d^\circ = 48^\circ\ 42'$ (sound being supposed to move over 1142 feet in a second). Ans. dist. . 5147.9

Second, when $a = 10$ sec, $b = 5$ sec, and $d^\circ = 60^\circ$ Ans. dist. . 9889·8

(10) (fig. 21′). From B the top of a ship's mast which was 80 feet above the water, the angle of depression* of another ship's hull at C upon the water was 20°: required the distance of the ships. Ans. 219.8.

Second, If the mast $= 143$ feet, and the angle of depression $= 35^\circ$. Ans. 204.2.

(11) (fig. 22). The side ABC of a building is an isosceles triangle, whose base BC is horizontal: an observer stationed at D directly opposite the middle point of BC and at 30 yards distance from it, finds the angle of elevation of A $44^\circ\ 1'\ 45''$, and the angle subtended by the extremities of BC $36^\circ\ 52'\ 15''$: required the sides of the triangle. Ans. $BC = 20$

$AB = AC = 30.67$.

* The angle between a horizontal line passing through the eye of the observer and a line drawn to the object is called the *angle of elevation* when the object is above the horizontal line, and the *angle of depression* when the object is below it: thus in fig. (21′) BCA is the angle of elevation of B above the horizontal line AC, and DBC is the angle of depression of C below the horizontal line BD.

(12) (fig. 23). Wishing to determine the height of an obelisk standing on the top of a declivity, I measured from its base a distance of a feet and then observed the angle formed by this line and a line drawn to the top to be $= \alpha^\circ$. Going on in the same direction b feet further the angle formed by the declivity with a line drawn to the top was $= \beta^\circ$. Required the height of the obelisk, when $a=40$, $b=60$, $\alpha^\circ=41^\circ$, and $\beta=23^\circ\ 45'$

Ans. height 57.62.

(13) (fig. 24). To determine the distance between two inaccessible objects C and D, a base AB on the same plane as the objects was measured and found to be 670 yards: the following angles were also observed: At A.. $BAD=40^\circ\ 16'$, $BAC=97^\circ\ 56'$. At B . $ABC=42^\circ\ 22'$, and $ABD=113^\circ\ 29'$.

Ans. $CD=1171.4$ yards.

(14) (fig. 24). To determine nearly the distance of two redoubts C and D, by which the entrance into a harbour is defended, a boat is placed at A with its head towards a distant object seen at E, and the angles $CAD=\alpha$ and $DAE=\beta$ were observed. The boat is then moved to B, a distance of a yards, directly towards E, and the angles $CBD=\gamma$ and $DBE=\delta$ are observed: required the distance CD.

If $a=1000$ yards, $\alpha=22^\circ\ 17'$, $\beta=48^\circ\ 1'$, $\gamma=53^\circ\ 15'$ and $\delta=75^\circ\ 43'$.

Ans. dist. 1290.

(15) (fig. 24). The distance between two objects C and D is known to be 6594 yards: on one side of the line CD, there are two stations A and B. At A the angles $CAD=85^\circ\ 46'$, $DAB=23^\circ\ 56'$. At B the

angle $CBD = 68° 2'$ and $CBA = 31° 48'$. Hence it is required to find the distance between the stations A and B. Ans. $AB = 4694$ yards.

(16) (fig. 25). To determine the height of an object EB on the top of an inaccessible hill, I took the angle of elevation ACE of the top of the hill $= a°$, and also ACB of the top of the object$=\beta$. Going then a yards in a direct line from the object and upon a horizontal plane, I found the angle of elevation of the top of the object $ADB=\gamma$: required the height of the object.

If $a=100$ yards, $a'=40°$, $\beta=51°$, and $\gamma=33° 45'$.

Ans. height$=46.67$.

(17) (fig. 26). Wanting to know the distance between two objects A and B, which could only be seen from a particular place D, I set up two staffs at C and E, and took the angles $ADC = 89°$, $ADB = 72° 30'$, and $BDE = 54° 30'$. I then measured DE and DC each$=200$ yards, and took the angles $BED=88° 30'$ and $DCA= 50° 30'$: required the distance AB.

Ans. 345.5 yards.

(18) (fig. 27). To determine my distance from an inaccessible object at O, without observing any angles, I measured a straight line $AB=500$ yards, from each extremity of which I could see O. I then measured from A and B in a direct line from O, AC and BD each$=100$ yards; finally I measured the diagonals AD and BC: the former was $= 550$ yards; the latter $= 560$ yards: required the distance of the object from A and B.

Ans. AO .. $=536.2$

BO .. $=500.47$

(19) (fig. 28). Wanting to know the distance AC of a hill from the station A and also its height OC, I measured a base $AB = 298$ yards, on ground nearly level, and at A and B observed with a sextant, the angles $BAO = 42^\circ.17'$ and $ABO = 79^\circ\ 29'$; and at A the angle of elevation $OAC = 4^\circ\ 51'$: required the distance AC and height OC. Ans. $AC = 344.6$, $OC = 29.2$.

(20) (fig. 29). Find how much the object B is elevated above a battery at A; from the following data: having observed the angle of elevation $BAC = 4^\circ\ 38'$, and measured $AC = 193$ yards to a point C, at C the angle $ACB = 76^\circ\ 32'$, and at A the angle $BAC = 45^\circ\ 4'$.

Ans. $BH = 17.8$ yards.

(21) (fig. 29). The angle of elevation of a tower 100 feet high, due north of an observer was 50°: what will be its angle of elevation after walking due east 300 feet.

Ans. $17^\circ\ 47'\ 45''$

(22) (fig. 30). The elevation of a balloon was observed to be 20° bearing N. E., and by another observer 4000 yards due south of the former, it was N.b.E.: required its height. Ans. 511.3

(23) (fig. 31). From a window which seemed to be on a level with the bottom of a steeple, I observed the angle ACB of elevation of the top of the steeple $= 40^\circ$: from another window of the same house 18 feet directly above the former, the angle EDB of elevation was $= 37^\circ\ 30'$: required the height and distance of the steeple.

Ans. height 210·4 dist. 250·8.

(24) (fig. 32). At B the top of a castle which stood on a hill near the sea-shore, the angle of depression

HBS of a ship at anchor was 4° 52′, and at *R*, the bottom of the castle, its depression *NRS* was 4° 2′: required the height of the top of the building above the level of the sea, supposing the castle itself to be 54 feet high: required also the horizontal distance of the vessel.

Ans. height 314·2 dist. 3690·3.

(25) (fig. 33). Wishing to know the breadth of a river, I measured a base of 500 yards in a straight line close to one side of it; and at each extremity of the base, I observed the angles subtended by the other end and a tree, standing on the opposite bank to be 53° and 79° 12′: required the breadth of the river.

Ans. breadth =529·5.

Second. If base = 108·3 and angles 43° 12′ and 60° 35′. Ans. breadth =66·49.

(26) (fig. 34). A base of 340 feet was measured on a sloping side of a hill, in a vertical plane passing through its summit. It made an angle of 10° 5′ with the horizontal plane: at the higher end of the base, the elevation of the hill was 46° 15′, and at the other end it was 40° 10′. The height of the eye at each observation was 5 feet: find the height of the hill above the lower end of the base. Ans. 1226 feet.

(27) (fig. 35). The distance between two stations *B* and *C* on a declivity is 220 yards. At *B*, the *horizontal* angle *Cba* between *C* and an object *A* on the top of a hill was found by *a theodolite* to be 70° 15′, and at *C* the horizontal angle *bCa* between *B* and *A* was 62° 33′: the vertical angles *ACa* = 32° 12′ and *BCb* = 8° 32′: find the horizontal distances of the object *A* from *C* and

B, namely, *Ca* and *ba*, and also the heights *Aa* and *AD* of the object above *C* and *B*.

Ans. hor. dist. of *A* from $C = Ca = 279.1$
.. .. $B = ba = 263.1$
perp. height of *A* above $C = Aa = 175\ 7$
.. .. $B = AD = 143.1$

(28) (fig. 35). Suppose, as in the last problem, that the direct distance between the stations *B* and *C* is 220 yards, also that at *C* the angle *BCb* (the elevation of *B*) $= 8° 32'$ and *ACa* (the elevation of *A*) $= 32° 12'$: but suppose now that *a sextant* is used in measuring the oblique angles, and that $ABC = 77° 8'$ and $ACB = 62° 18'$. Find as before the *horizontal* distances between *A* and the stations *B*, *C*, and the heights of *A* above *B* and *C*. Ans. see last problem.

(29) (fig. 36). Wanting to know the distance of two objects *A* and *B* from each other, and from another object *D*, all in the same plane, on *BA* produced on the side of *A*, a point *C* was taken and *CD* being measured was found to be 549.4 yards, and the angle $C = 57°$. At *D* the angle *CDA* was observed to be 14°, and the angle $BDA = 41° 30'$: required the distance of *A*, *B*, and *D* from each other.

Ans. $AB = 349.52$
$AD = 487.27$ and $BD = 349.52$

(30) (fig. 37). Wishing to find the distance of a battery at *B*, from a fort at *F*, which cannot be seen from the battery in consequence of the ground between *B* and *F* being covered with wood, &c., I measured distances *BA* and *AC* to points *A* and *C* where both the

fort and battery were visible, the former being 2000 yards and the latter 3000: and observed the angles BAF $=34^\circ\ 10'$, FAC $= 74^\circ\ 42'$ and FCA $= 80^\circ\ 10'$. From these data it is required to find the distance of the fort from the battery. Ans. 5422.

(31) (fig. 38). Coming from sea, at the point D, I observed two headlands A and B, and inland at C a steeple which appeared between the headlands, I found from a map that the headlands were 5.35 miles from each other; that the distance from A to the steeple was 2.8 miles and from B to the steeple 3.47 miles: I observed with a sextant the angle ADC $= 12^\circ\ 15'$ and BDC $= 15^\circ\ 30'$: required my distance from each of the three objects.

Ans. AD $= 11{\cdot}26$, CD $= 12{\cdot}46$, BD $= 11{\cdot}03$.

(32) (fig. 38). Three stations of the Trigonometrical survey can be seen from the Eddystone Light-house (*D*) Carraton hill (A), Kitts hill (C), and Butterton hill (B). It appears from the survey (vol. 1, page 375) that the distance between the stations A and B is 131576 feet, between C and B, 100969 feet, and between A and C 33427 feet; it is found also from the data given in page 402, that at the light-house the angle ADC $= 15^\circ\ 15'\ 55''$ and CDB $= 48^\circ\ 45'\ 53''$: required the distance of the light-house from each station.

Ans. 123411, 126896, 121123.

(33) (fig. 39). Required the distance of the three objects A, B and C from the point D, situated *within* the triangle, from the following data: AB $= 267$, AC $= 346$, BC $= 209$, angles ADC $= 128^\circ\ 40'$, and ADB $= 91^\circ\ 20'$. Ans. 189.33, 178.85, 104.05.

The following nine problems require a knowledge of the compass.

(34) (fig. 40). A headland C bore due North of a ship at A; after sailing 10 miles due East to B, the headland bore N. W.: required the distance of the headland at each observation. Ans. 10 and 14.14.

(35) (fig. 41). A fort A bore from a ship C due North, by compass, and after sailing N. E. by compass $14\frac{1}{2}$ miles to B the fort bore N. W.: find the distance of the fort at both observations. Ans. 20·5 and 14·5.

(36) (fig. 42). A boat is placed at A due West of a ship at B; after sailing N. W. 10 miles to C, the boat bears S.S.W.: required the distance of the boat from the ship at the first observation. Ans. 10.

(37) (fig. 43). Sailing along a coast a headland C was observed to bear N.E.b.N; having run E.b.N. 15 miles to B, the headland bore W. N. W.: find the distance from the headland at each observation.

Ans. 8.496, 10.81.

(38) (fig. 44). A cape C was observed to bear from us N. W., and another headland H to bear N.N.E $\frac{1}{2}$E.; standing away E.N.E.$\frac{1}{2}$E. 23 miles to B, we found the first bore from us W.N.W., and the second N.b.W.$\frac{1}{2}$W.: required the distance and bearing of the cape from the headland. Ans. W.$\frac{1}{4}$S. . . 42.33.

(39) (fig. 45). A church C bears from a battery B, E.N.E. 960 yards; how must the church bear from a ship at sea, supposing her to run in until the battery is North 2000 yards. Ans. N. 20° 32′ E.

(40) (fig. 46). A cape C bears from a headland H,

W. $\frac{1}{4}$S. 4.23 miles: how must the cape bear from a ship which runs in towards the headland on a N.b.W. $\frac{1}{2}$W course, until the headland is 2·3 miles distant from the ship. Ans. W. N. W.

(41) (fig. 47). A ship S was 2640 yards due South of a light-house AB, and after sailing N.W.b.N. 800 yards to D, its angle of elevation BDA was 5° 25′: required its height. Ans. 192 yards.

(42) (fig. 48). The bearings of two objects A and B in the same latitude from a ship at C are N.N.E. and N.E.b.E., and the distance from A is 10 miles: required the distance from B. Ans. 16.63.

(43) (fig. 49). Find the angle which the *line of metal* makes with the axis produced in a piece whose dimensions are known. Ex. Diameter of breech = 12.44 inches; of muzzle 9.84 inches, length of gun = 77$\frac{1}{4}$ inches.

Ans. 0° 57′ 51″.

Calculations of a front of Fortification.

(44) (fig. 50). Let AB be the exterior side of a regular pentagon, and suppose the side to be 350 yards, the perpendicular DE 50 yards, the face AH of the bastion 100 yards, and the line of defence AG equal to AK. As the length and position of the other lines depend on these values, it is required to find the magnitude and position of the following parts.

(1) Angle of the tenaille AEB .... Ans. 148° 6′ 30″
(2) Line of defence AK or AG 255.33
(3) Flanked angle $HAH_{\prime}$ .... 76° 6′ 30″
(4) Curtain FG .... Ans. 141.02

(5)	Angle of the shoulder AHF..........	117° 0′
(6)	Length of flank FH.................	43.5
(7)	Flanked angle GFH..................	101° 3′

(45*a*) (fig. 51). A maypole was broken by the wind, and its top struck the ground 20 feet from its base, and being again fixed was broken a second time 5 feet lower, and its top extended 10 feet farther: required its height.
Ans. 50 feet.

(46) (fig. 52). The summit A of a hill bore due East of a spectator at B, and E.N.E. of a spectator at a point C due south of B; the elevation of the point A at B was 20°: required its elevation at C.

Ans. 18° 36′.

(47*a*) (fig. 53). If the base of an oblique angled triangle be 40, and the other two sides 20 and 30; what is the length of the perpendicular from the vertical angle.
Ans. 14.523.

(48) (fig. 54). The elevation of a spire DC at one station A was 23° 50′ 15″, and the horizontal angle at this station, between the spire and another station B was 93° 4′ 20″; the horizontal angle at B was 54° 28′ 30″, and the distance AB between the two stations was 416 feet: required the height of the spire.

Ans. 278.7.

(49*a*) (fig. 52). An observer finds the angle of elevation of a tower at a point B, to be 23° 18′; after walking from B, 300 feet in a direction at right angles to the line joining B with the foot of the tower, the angle of

elevation was 21° 16′: required the height of the tower, and its distance from A.

Ans. height 272·7 feet, dist. 633·4.

(50*a*) (fig. 55). On the bank of a river stands a column 200 feet high, on which is a statue 50 feet high, and to an observer on the opposite bank the statue subtended an equal angle with a man 6 feet high, standing at the base of the column; required the width of the river.

Ans. 82·5.

(51*a*) (fig. 56). A flag-staff 12 feet high, on the top of a tower, subtended an angle of 48′ 20″ to an observer at the distance of 100 yards from the foot of the tower: required the height of the tower. Ans. 401.4 feet.

(52) Find the area of a triangle whose base is 40 feet, and perpendicular 30 feet.

Ans. $66\frac{2}{3}$ square yards.

(53*a*) If from a right angled triangle whose base is 12 and perpendicular 16 feet, a line be drawn parallel to the perpendicular cutting off a triangle whose area is 24 feet: required the sides of the triangle.

Ans. 6, 8, 10.

(54*a*) Required the side of an equilateral triangle, the area of which is 180 square yards. Ans. 20.389 yds.

(55*a*) Given the base of a triangle equal to 476.25 yards, and the angles at the base 27° 10′ 15″, and 35° 10′ 15″: to find the area. Ans. 33680.

(56*a*) The area of a triangle is 6, and two of its sides are 3 and 5: find the third side.

Ans. 4 or $\sqrt{52}$.

(57*a*) (fig. 57). The straight line EF is drawn parallel to the base of the triangle ABC whose altitude is 10 feet: find the distance of EF from the base BC, so that it may divide the triangle into two equal parts.

Ans. Dist. from base $= 2.929$ feet.

(58*a*) When a parish was inclosed, the allotment of one of the proprietors consisted of two pieces of ground, one of which was in the form of a right angled triangle; the other was a rectangle, one of the sides of which was equal to the hypothenuse of the triangle, the other to half the greater side; but wishing to have his land in one piece, he exchanged his allotments for a square piece of ground of equal area, one side of which equalled the greater of the sides of the triangle which contained the right angle. By the exchange, he found he had saved 55 yards of paling: what are the area of the triangle and rectangle.

Ans. Triangle $= 181.5$, Rectangle $= 302.5$

(59*a*) The area of a right angled triangle whose sides are in arithmetic progression $= 216$: determine the sides.

Ans. 18, 24, 30.

(60*a*) What is the side of that equilateral triangle, whose area cost as much paving, at 8d. per foot, as the pallisading the three sides did at 7s. a foot.

Ans. 72.74 feet.

(61*a*) Given a side a of a regular polygon of n sides, to find the area.

Ans. Area $= \dfrac{na^2}{4} \cot. \dfrac{180^\circ}{n}$

Ex. (1) An eight sided polygon or octagon whose side = 16 yards. Ans. area = 1236·1

(2) A ten sided polygon, or decagon, whose side $= 20\frac{1}{2}$ yards. Ans. area = 3233·5.

(62*a*) Given the area A of a regular polygon of n sides: to find a side a.

$$\text{Ans. } a = \sqrt{\frac{4A}{n} \tan. \frac{180}{n}}$$

Ex. (1) The area of a regular octagon is 1236·1 square yards, find a side. Ans. side = 16 yards.

(2) The area of a decagon is 3233·5 square yards: find a side. Ans. side = 20.5 yards.

(63*a*) (fig. 58). To make a regular polygon of n sides equal to a given triangle ABC. Take $AD = \frac{AB}{n}$: draw AG making angle $CAG = \frac{360^\circ}{n}$: draw DE parallel to AC, and take AF = a mean proportional to AC and AE: then AF is the radius of a circle that will contain the required polygon: required a proof.

(64) A ship sailing on a S. W. course, bore from me due south, and the angle subtended by the head and stern was 20° 15′, and her length was known to be 160 feet: required her distance. Ans. 1.94 miles.

(65*a*) A ship sailed S. x° W., and met another ship which had sailed N. $(x + 10)$ W from the same meridian; the distances sailed were as 3 : 2, and their distance from the meridian left was 100 miles: required the difference of latitude. Ans. 479.8 miles.

(66*a*) How far may the surface of the sea be seen by

a man standing 6 feet above it, the radius of the earth being 4000 miles. Ans. 5307 yards.

(67*a*) From the top of a mountain, 3 miles high, the true depression of the horizon was found to be $2^\circ\ 13'\ 27''$: required the diameter of the earth, supposing it to be a sphere. Ans. 7952 miles

(68*a*) If at the top of a mountain, the true depression of the horizon of the sea is found to be $1^\circ\ 31'$: what is the height of the mountain supposing the earth to be a sphere whose diameter is 8000 miles. Ans. 1.402 miles.

(69*a*) From a station on the side of a river whose banks are parallel, and width 560 yards, the angle formed by two objects on the opposite bank was $35^\circ\ 10'$, and their distance from each other 400 yards: required their distance from the station. Ans. 560.1 and 694.3.

(70*a*). Walking along a road, I observed the elevation of a tower AB to be 20°, and the angular distance of its top from an object in the road to be 30°: the nearest distance of the tower from the road was 200 feet: required its height. Ans. 187.5.

(71*a*) One angle of a triangle is $129^\circ\ 34'$ and the two sides about that angle are to each other in the proportion of 4 to 7: required the other two angles.

Ans. $32^\circ\ 11'\ 7''$ and $17^\circ\ 44'\ 53''$.

(72*a*) The three sides of a plane triangle $=6$, and the angles are to each other as 1, 2, 3: find the sides.

Ans. 1·268, 2·536, 2·196.

(73*a*) The perimeter of a triangle is equal to 100 yards, and the angles are to each other in the proportion of 1,

2 and 4. It is required to find the sides of the triangle.
Ans. 19·8, 35·69, 44·51.

(74*a*) The perimeter of a right angled triangle = 24 yards and one of the angles = 30°: find the sides.
Ans. 5·072, 8·784, 10·144.

(75*a*) In a plane triangle ABC given A=80° and $a = 400$: required the other sides their sum being 600.
Ans. 369·25, 230·75.

(76*a*) In a plane triangle ABC given A=60° c=400 and the sum of the other two sides a and b=600: required the sides a and b. Ans. 250 and 350.

(77*a*) The perimeter of a right angled triangle is 24 feet, and its base is 8 feet: find the other sides.
Ans. 6 and 10.

(78*a*) At 80 feet distant from a steeple, the angle made by a line drawn from its top to the place was double to that made by a line drawn from the top to a point 250 feet from the steeple on the same level: required the height of the steeple. Ans. 150.

(79*a*) Given the base a, the vertical angle A, and the sum of the sides of the plane triangle ABC=b: to find the sides x and y.

$$\text{Ans. } x+y=b, \text{ and } xy=\tfrac{1}{2}\,(b+a)\,(b-a)\,sec.^2\frac{A}{2}$$

from which equations, x and y may be found.

(80*a*) Given the base a, the vertical angle A. and the difference of the sides of a plane triangle $=d$: to find the sides x and y.

Ans. $x-y=d$, and $xy=\frac{1}{4}\,(a+d)\,(a-d)$ cosec.^{2}

(81*a*) Given the base a, the difference of the angles x and y at the base $= d$, and the sum of the two other sides of the plane triangle $=b$: to solve the triangle.

Ans. $\text{Cos.}\ \frac{1}{2}(x+y) = \frac{a}{b}\cos.\ \frac{1}{2}\ d$, which determines the sum of the unknown angles, and thence, with their difference already known, the angles x and y.

(82) Given the base a, the difference of the angles x and y at the base $=$ D, and the difference of the sides of a plane triangle $= d$: to solve the triangle.

Ans. $\sin.\ \frac{1}{2}(x+y) = \frac{a}{d}\sin.\ \frac{1}{2}\ \text{D}$, and $x - y$ is already known.

(83) Given the base of a triangle b, and one of the angles at the base A, and the difference of the other sides $= d$: to solve the triangle.

Ans. $\text{Tan.}\ \frac{1}{2}\ \text{C} = \frac{b-d}{b+d}\tan.\ \frac{1}{2}\ \text{A}$, from which the angle C is found: and thence the other parts of the triangle.

(84*a*) Given the base of a plane triangle b, one of the angles at the base A, and the sum of the other sides $= m$: to solve the triangle.

Ans. $\text{Tan.}\ \frac{1}{2}\ \text{C} = \frac{m-b}{m+b}\cot.\ \frac{1}{2}\ \text{A}$.

(85*a*) Given the angles and the perimeter of a plane triangle: to find the sides.

Ex. perimeter $= 100$ yards, $\text{A} = 102^\circ\ 51'\ 30$, $\text{B} = 25^\circ\ 42'\ 45''$ and $\text{C} = 51^\circ\ 25'\ 45''$.

Ans. $a = 44{\cdot}51$, $b = 19{\cdot}8$, $c = 35.69$.

(86*a*) Given the distances between three stations in a straight line with an object standing upon a horizontal plane: and the angles at the points E, D, C, $= \theta$, $90-\theta$ and 2θ in order: θ being unknown: to find its height. *Ex.* let ED $=20$ and DC $=20$ Ans. 38.73.

(87) Investigate analytical expressions for calculating the distance of a station from each of three points: having given the distances of the points from each other and the angles which they subtend at the station.

Ans. (fig. 38). Let D be the station A, B, C, the three points, CDB$=\alpha$, ADC$=\beta$, CB$=a$, CA$=b$, ACB$=$C, CAD$=x$ and CBD$=y$: then $x+y = 360° - (C+\alpha+\beta)$ and

$$\tan \tfrac{1}{2}(x-y) = \frac{1-\dfrac{b \sin. \alpha}{a \sin. \beta}}{1+\dfrac{b \sin. \alpha}{a \sin. \beta}} \tan. \tfrac{1}{2}(x+y),$$

from these equations, the angles x and y may be determined, and thence the distance of the station from each of the three points. For a numerical example see prob. (31).

ASTRONOMICAL AND NAUTICAL PROBLEMS.

Spherical Trigonometry derived its origin from the computations necessary in astronomy, and its principal applications are still furnished by the same science.

To assist the student in understanding the nature of the following problems we will give in this place a few astronomical terms and definitions.

The astronomer conceives all the heavenly bodies to be contained within a hollow sphere of great but indefinite magnitude: he supposes this sphere (the interior surface of which is called the *celestial concave,*) to have the same center as the earth; and in all applications o spherical trigonometry, he employs, instead of the real position of any of the heavenly bodies, the point in which the celestial concave is cut by a straight line drawn through the *centers* of the earth and of the body: this point is called the *true* place of the heavenly body. Thus (fig. *a*) let $QZPQ_,P_,$ represent the celestial concave, *m* a heavenly body, C the center of the earth, join *Cn* and produce it to cut the celestial concave in $M_,$: then $M_,$ is the true place of the heavenly body *m*. Let A be the place of a spectator on the surface of the earth:

join Am and produce it to the celestial concave at M: then M is called the *apparent* place of m.

The extremities of the axis on which the celestial concave appears to revolve, in consequence of the earth's diurnal motion are called the *poles of the heavens*: thus, $PP_{\prime}$ represents the axis of the heavens, P and $P_{\prime}$ are the poles: the points p and $p_{\prime}$ in which the earth's surface $pqp_{\prime}q_{\prime}$ is cut by this axis are called the *poles of the earth*: that great circle on the surface of the earth which is equidistant from each of its poles, is called the *terrestrial equator*: in the fig., $qq_{\prime}$ represents the plane of the terrestrial equator. The terrestrial equator extended to the celestial concave, as $QQ_{\prime}$ forms the *celestial equator*. A circle touching the earth where the spectator stands, and extending to the celestial concave is called the *visible* horizon, and a circle parallel to the visible horizon which passes through the center of the earth and extends to the celestial concave is called the *rational* horizon: thus hr represents the visible, and HR the rational horizon of the spectator at A. These two circles however form one and the same circumferance on the celestial concave, thus the points R and r in the figure may be supposed to coincide: this may be readily conceived when we consider that the distance of any two points on the surface of the earth will make no sensible angle at the celestial concave: therefore, either of these two circles is to be understood by the word *horizon*. Of the poles of the horizon of any place, that which is over the place is called the *zenith*, and the other the *nadir*, as z and $z_{\prime}$ in (fig. a).

Great circles passing through the zenith are called *circles of altitude* or *vertical circles*: the circle of altitude passing through the pole of the heavens is called the *celestial meridian*: the points of the horizon through which the celestial meridian passes are called the north and south points, and a circle of altitude at right angles to the meridian is called the *prime vertical*, this last circle cuts the horizon in the east and west points.

These circles will be more clearly understood by means of (fig. *b*), in which N.W.S.E. represents the horizon of a spectator whose zenith is supposed to be Z: P is one of the poles of the heavens, and N.S.E.W. the north, south, east and west points of the horizon. NZS is a circle of altitude passing through the zenith and pole, and therefore represents the celestial meridian: ZD ZO &c. are circles of altitude, and WZE is the prime vertical.

Since the horizon and celestial equator are both perpendicular to the celestial meridian, the points where the horizon and celestial equator intersect each other must be 90° distant from every part of the meridian, that is the celestial equator cuts the horizon in the east and west points: draw the curve EQW (fig. *b*) to cut the horizon in the east and west points this will represent the celestial equator. Since the poles of the heavens are 90° distant from the equator, QP = 90°. P is called the elevated pole, or the one above the horizon.

The *distance* ZQ fig. (*a*) and (*b*) of the zenith from the equator represents the *latitude of the spectator*: this may be more clearly seen in fig. (*a*) where PZQH

represents the plane of the celestial meridian, Z the zenith of the spectator at A; *hr* or HR his horizon, P the elevated pole, $QQ_{\prime}$ drawn at right angles to $P_{\prime}P$ represents the plane of the celestial equator, and the angle ZGQ (the earth being considered as a spheroid) is the latitude of the spectator: but since the distance ZQ of the zenith from the equator, measures the same angle, this arc is taken to represent the latitude* ZP fig. *(a)* and *(b)*, the complement of ZQ′ is called the *colatitude*.

In consequence of the earth's motion in its orbit round the sun; the latter body appears to move eastward in the celestial concave, and, in the course of a year to describe, among the fixed stars, a great circle; this circle is called the *ecliptic*. The equator and ecliptic are inclined to each other at an angle of about 23° 28′, called the *obliquity* of the ecliptic. In fig. (*b*) ACT represents the ecliptic and the angle MAR the obliquity. The points in which the equator and ecliptic intersect are called the *first point of Aries* and *first point of Libra*: the former being the point where the sun crosses the equator northward is called the *vernal equinoctial point*, and the latter the *autumnal* equinox. Great circles passing through the poles of the heavens are called circles of *declination*, and great circles passing through the poles of the ecliptic are circles of *latitude*: thus in fig. *b*, PR, PA, &c. are circles of declination and if $P_{\prime}$

* If CA be joined and produced to cut the celestial concave in $Z_{\prime}$ the arc $Z_{\prime}Q$ is called the latitude *on the sphere*, or the reduced latitude.

represent the pole of the ecliptic, PM is a circle of latitude.

The *declination* and *right ascension* of a heavenly body may be defined thus: the *declination* of a heavenly body is the arc of the circle of declination passing through its place in the celestial concave, intercepted between this place and the celestial equator: the *right ascension* of a heavenly body is the arc of the celestial equator intercepted between the first point of Aries and the circle of declination passing through the place of the body in the celestial concave. Or it is the angle at the pole of the heavens between the circles of declination passing through the first point of Aries and the place of the heavenly body: the arc or angle being measured round the equator or the pole from west through south to east or in the direction of the earth's motion round the sun: thus in fig. *b*, if X be a heavenly body, and PXR a circle of declination passing through its place, XR is its declination, and AR or the angle APR its right ascension. In like manner if a circle of latitude be drawn through any point in the celestial concave, the part of it between the point and the ecliptic is called the *latitude* of the point, and the arc of the ecliptic extending eastward from the first point of Aries to the circle of latitude is called the *longitude* of the point: thus the latitude of X is XM and longitude AM. The *altitude* of a heavenly body is the arc of a circle of altitude intercepted between the place of the body and the horizon: thus XO is the altitude of X. The *azimuth* of a heavenly body is the arc of the horizon intercepted between the north or south

points, and the circle of altitude passing through the place of the body; or it is the corresponding angle at the zenith between the celestial meridian, and the circle of altitude passing through the body: thus SO or NQ, or the angles OZS or QZN is the azimuth of X. The *amplitude* of a heavenly body is the distance from the east at which it rises, or the distance from the west at which it sets, these arcs or distances being measured on the horizon: thus the amplitude of X is the arc WD or ED_2 (the dotted line D_2XD being the arc of a *parallel* of declination described by X from rising at D_2 to setting at D.) The *hour angle* of a heavenly body is the angle at the pole between the celestial meridian and the circle of declination passing through the place of the body: thus ZPX is the hour angle of X. If L be the place of the sun, *west* of meridian, its hour angle ZPL is called *apparent* time: but when the sun is *east* of the meridian as at C, then apparent time is found by subtracting the hour angle ZPC from 24 hours.

If Y be the place of any heavenly body when on the prime vertical, V its place when its hour angle ZPV is 6 hours, and D its place when setting, then if we put the

latitude of spectator ZQ .. .. $=l$
hour angle ZPD when body is setting . ..... $=h$
hour angle ZPY when body is due west .. .... $=h_1$
altitude VD, at six o'clock . ... $=a$,
altitude WY when due west . ... $=a_1$
azimuth ND from north at setting .... $=Z$
amplitude WD .... $=m$
azimuth PZV at six o'clock.... .... $=Z_1$

and the declination of the heavenly body .... $= d$
the following formulæ are easily deduced from Napier's Rules for quadrantal and right angled triangles.

(1) $\cos. h = -\tan. l \tan d$ (4) $\cos. l = \cot. d \cot. Z_{,}$
(2) $\sin. d = \cos. l \cos. Z$ (5) $\sin. d = \sin. l \sin. a_{,}$
$= \cos. l \sin. m$ (6) $\cos. h_{,} = \cot. l \tan. d.$*
(3) $\sin. a = \sin. l \sin. d$

(88) Given the sun's meridian altitude = 70° (zenith north of the sun) and its declination 20° N.: required the latitude of the place. Ans. 40° N.

(89) Given the sun's meridian altitude = 70° (zenith north), and its declination = 5° S: required the latitude. Ans. 15° N.

(90) Given the sun's meridian altitude 70° (zenith south), and declination 25° N.: required the latitude. Ans. 5° N.

(91) Given the sun's meridian altitude = 30° (zenith south), and declination 10° N.: required the latitude. Ans. 50° S.

(92) Given the altitudes of a circumpolar star at its inferior transit = 20°, and at its superior transit = 70°: required the latitude. Ans. 45° N.

(93) A circumpolar star passes the zenith of a place and its altitude at the inferior transit is 20°: required the latitude. Ans. 55° N.

(94) The altitude of a circumpolar star at its inferior

* In these formulæ, both the latitude and sun's declination are of the same name. In the problems, these quantities are considered north, unless the contrary is expressed.

transit is equal to its zenith distance at its superior transit: required the latitude. Ans. 45° N.

(95) Given the sun's altitude (a), and hour angle (h) when the declination is nothing: to find the latitude (l).

Ex. $a=22° 56'$, $h=3^h$. Ans. $l=56° 33' 30''$ N.

(96) Given the sun's amplitude m, and declination d: to find the latitude l.

Ex. $m=$E $37° 30'$ N., $d=15° 12'$ N.

Ans. $l=64° 29' 15''$ N.

(97) Given the sun's altitude a at six o'clock, and declination d: to find the latitude.

Ex. $a=18° 15'$, $d=20° 4'$ N. Ans. $l=69° 31' 40''$ N.

(98) Given the latitude l, and sun's declination d; to find his altitude and azimuth at six o'clock.

Ex. $l=50° 48'$ N., $d=23° 27' 45''$,

Ans. $a=17° 58' 15''$, az.$=$N $74° 39' 30''$ E.

(99) Given the sun's declination d, and latitude of the place l: to find his altitude and the time when he is on the prime vertical.

Ex. $d=23° 27' 45''$ N, $l=50° 48'$.

Ans. Alt. $= 30° 55'$, time $= 4^h 37^m 4^s$.

(100) Given the latitude of the place, and the sun's declination, to find his amplitude, time of rising or setting, and the length of the day and night.

Ex. $l=50° 48'$ N. $d=18° 28'$ N.

Ans. Ampl.$=$E. $30° 4' 30''$N.; sun rises at $4^h 23^m 19^s$ A. M.; length of day $=15^h 13^m 22^s$.

(101) The altitude of a star when due east was 20° and it rose E.b.N.: required the latitude.

Ans. Lat.$=29° 42'$ N.

(102) The altitude of a star whose declination was 20° N. when due west was 30°: required the latitude.

Ans. Lat. $=43^\circ\ 9'\ 15''$ N.

(103) Given the sun's altitude (west of meridian) and declination, and the latitude of the place: to find the azimuth and the apparent time of observation.

Ex. $a=46^\circ\ 20'$, $d=23^\circ\ 27'\ 45''$, $l=50^\circ\ 48'$.

Ans. Az. $=$ N. $111^\circ\ 51'$ W; app. time $=2^h\ 57^m\ 16^s$.

(104) Given the sun's altitude, declination and azimuth: to find the latitude.

Ex. $a=18^\circ\ 45'$, $d=22^\circ\ 10'$ N.; az. $=$ S. $57^\circ\ 45'$ W.

Ans. Lat $=59^\circ\ 4'$ N.

(105) Given the sun's altitude (a), hour angle h, and decl. d: to find the latitude (zenith being north of the sun).

Ex. $a=37^\circ\ 20'$, $h=2^h\ 15^m$, $d=22^\circ 30'$ N.

Ans. Lat. $=71^\circ\ 31'$ N.

(106) Given the right ascension of the sun, RA, and obliquity of the ecliptic ω: to find his longitude and declination.

Ex. (1). R. A. $=4^h\ 10^m\ 20^s$, $\omega=23^\circ\ 27'\ 45''$.

Ans. Long. $=64^\circ\ 33'\ 15''$, $d=21^\circ\ 4'\ 15''$ N.

Ex. (2). R. A. $=17^h\ 10^m$, $\omega=23^\circ\ 27'\ 45''$.

Ans. Long. $=258^\circ\ 30'\ 15''$, $d=22^\circ\ 58'$ S.

(107) Given the right ascension and declination of a heavenly body, and the obliquity of the ecliptic ω: to find its latitude and longitude.

Ex. (1). R.A. $=2^h\ 59^m\ 37^s$, $d=21^\circ\ 27'\ 48''$ N $\omega=23^\circ\ 27'\ 45''$.

Ans. Lat. $=4^\circ\ 15'$ N., long. $=48^\circ\ 37'\ 30''$.

Ex. (2). R. A. = $16^h\ 14^m$, $d = 25°\ 51'$ N., $\omega =$ $23°\ 27'\ 45''$.

Ans. lat. = $46°\ 6'\ 15''$ N., long. = $234°\ 36'\ 30''$.

(108) Two places have the same latitude, namely, 45° N., and their difference of longitude is $10°\ 36'$: required their distance.

Ans. 449.25 nautical miles.

(109) Required the distance from Portsmouth to Buenos Ayres: lat. of Portsmouth $50°\ 48'$ N., long. $1°\ 6'$ W: lat. Buen. Ayres $34°\ 37'$ S. long. $58°\ 24'$W.

Ans. 5949.8 nautical miles or 6847.2 Eng. miles ($69\frac{1}{20}$ to a degree).

(110) Required the distance of the moon from *a* Leonis (Regulus): the right ascen. and decl. of the former being $0°\ 32'\ 45''$ and $5°\ 19'$ S., and of the latter, $148°\ 18'\ 45''$, and $13°\ 10'\ 15''$ N. Ans. 148° 2

(111) Required the sun's azimuth and depression below the horizon at 7^h P. M. (app. time) the decl. being $10°\ 15'$ S., and the latitude of the place $50°\ 48'$ N.

Ans. $17°\ 23'\ 30''$.

(112) Given the sun's longitude $202°\ 24'\ 15''$, and the moon's latitude and longitude $4°\ 54'\ 30'$ N. and $69°\ 25'\ 30''$: required their distance.

Ans. $112°\ 53'\ 30''$.

(113) If a ship from latitude $50°\ 10'$ N., start on a S. W. course, and sail 100 miles on a great circle: what will be her last course. Ans. S. $45°\ 38'$ W.

(114) The latitude of a place A is 40° N. of B 50° N. and their distance from each other 20°: the longitude of A is 15° E.: required the latitude and longitude of

another place C to the north of, and 20° distant from A and B.

Ans. Lat. 59° 37′ N. Long. 21° 13′ E. or 8° 47′ E.

(115) The distance of a comet from Aldebaran and Regulus were observed to be 40° 12′ and 51° 36 respectively: required its latitude and longitude; the respective latitudes of the two stars being 5° 28′ 45″ S., and 0° 27′ 30″ N. their longitudes 67° 12′ 15″ and 147° 15′ 30″, and the comet being south east of the arc of a great circle joining them.

Ans. Lat. 28° 0′ 15″ S., long. 102° 19′.

(116) Required the beginning of morning and the end of evening twilight, at a place in latitude 54° 36′ N.; the sun's declination being 8° 30′ N., (twilight being supposed to begin and end when the sun is 18° below the horizon. Ans. 2^h 46^m A. M., 9^h 14^m P.M.

(117) Suppose two altitudes of the sun observed in the forenoon in the same place, at the interval of an hour and a half to be 28° 40′ and 39° 50′: required the latitude of the place, the declination being 23° 26′ N. at both observations. Ans. Lat. 59° 17′ 15″ N.

(118) The altitudes of *a* Hydræ and Regulus were observed at the same time to be 40° 44′ and 45° respectively, the right ascension and declination of the former being 9^h 16^m and 7° 37′ S., of the latter 9^h 53^m and 13° 9′ N.: required the latitude.

Ans. Lat. = 26° 37′ N.

(119*a*) The difference of longitude between two places in the same latitude, namely, 33° 51′ S. is 136° 10′: how much shorter is the distance between them on the arc

of a great circle, then on their common parallel: and what is the highest latitude attained by a ship sailing from one to the other on the arc of a great circle.

Ans. Difference of distances $=737.6$ English miles: highest lat. $=60^\circ\ 54'$ S.

(120) What is the highest latitude attained by a ship sailing on the arc of a great circle from Port Jackson to Cape Horn, their latitudes being $33^\circ\ 51'$ and $55^\circ\ 58'$, and the difference of longitude $140^\circ\ 27'$. Ans. $72^\circ\ 41'$.

(121) Required the sun's azimuth and depression below the horizon at 7^h P. M.: when the declination was $10^\circ\ 15'$ S., and the latitude of the place $50^\circ\ 48'$ N.

Ans. Az. $=$ N. $84^\circ\ 53'$ W.; depr. $= 17^\circ\ 24'$.

(122) Determine the bearing or azimuth of the two stars Aldebaran and Pollux when on the same vertical circle; the lat. of place being 25° N.: the R. A. and declination of former star being $4^h\ 26^m\ 46^s$ and $16^\circ\ 11'$ N.; of the latter $7^h\ 35^m\ 13^s$ and $28^\circ\ 24'\ 26''$ N.

Ans. S. $75^\circ\ 5'$ W.

(123) In a certain latitude (zenith N.) the moon's true altitude was $18^\circ\ 2'\ 30''$ (East of meridian) in the same vertical circle with a star whose right ascension and declination were $9^h\ 59^m\ 42^s$, and $12^\circ\ 45'\ 45''$ N: the moon's R. A. and declination being $12^h\ 35^m\ 54^s$, and $1^\circ\ 42'\ 30''$ S: required the latitude.

Ans. $19^\circ\ 55'\ 30''$ N.

(124) Given the latitude of the place and the sun's declination to find the time when he is due west, and the altitude. *Ex.* Lat. $=48^\circ$ N., decl. $=20^\circ$ N.

Ans. Hour angle $=4^h\ 43^m\ 28^s$, alt. $=27^\circ\ 24'$.

(125) Given the latitude of the place l and the sun's declination: to find his amplitude and hour angle when setting. *Ex.* $l = 48°$ N., decl. $= 20°$.

Ans. Ampl. $=$ W. $30° 44' 30''$ N., $h = 7^h 35^m 22^s$.

(126*a*) Given the altitude of the sun when due west, and at 6 o'clock: to find the latitude and declination.

Ex. Alt. when west $= 27° 24'$; at $6^h = 14° 43' 30''$.

Ans. $l = 48°$ N., decl. $= 20°$ N.

(127*a*) Given the sun's altitude at 6 o'clock, and its amplitude: to find the latitude and declination.

Ex. Alt. at $6^h = 14° 43' 30''$, ampl. $= 30° 44' 30''$.

Ans. Lat. $= 42°$ or $48°$; $d = 22° 20'$ or $20°$.

(128*a*) Given the sun's altitude at 6 o'clock, and the hour angle when setting: to find the latitude and decl.

Ex. Alt, at $6^h = 14° 43' 30''$ $h = 7^h 35^m 22^s$.

Ans. Lat. $48° 1'$ and decl $= 20°$.

(129*a*) Given the times at which the sun sets and is west on the same day, at a particular place; to find the latitude of the place, and the declination.

Ex. Hour angle when W. $= 4^h 43^m 28^s$ setting $= 7^h 35^m 22^s$. Ans. $l = 48°$ N., declination $= 20°$ N.

(130*a*) Given the sun's declination and the interval between the times at which he is west and sets: to find the latitude.

Ex. Decl. $= 20°$ N., interval $= 2^h 51^m 54^s$.

Ans. $l = 48°$ N. or $42°$ N.

(131*a*) Given the amplitude of the sun, and the azimuth at 6 o'clock: to find the latitude and declination.

Ex. Ampl. $=$ W. $30° 44' 30''$ N., azimuth $=$ N. $76° 18' 45''$ W. Ans. $l = 48°$ N. and $d = 20°$ N. nearly.

(132*a*) Given the sun's meridian altitude, and his altitude at 6 o'clock; to find the latitude and declination

Ex. Mer. alt. $=62°$, alt. at $6^h = 14° \ 43' \ 30''$.

Ans. $l = 48° \ 0' \ 20''$ N., $d = 20°$ N.

(133*a*) Given the interval between the times at which the sun is west and sets at a place whose latitude is known: to find the declination

Ex. Lat. 48° N., interval $=2^h \ 51^m \ 54''$.

Ans. Decl. $=20°$ N.

(134*a*) At a given place, to find the greatest azimuth of a heavenly body, whose declination is greater than the latitude of the place: to find also the time and altitude on a given day, when the heavenly body will have the greatest azimuth, and when consequently it will appear to move perpendicularly to the horizon.

Ex. In lat. 20 N. when the sun's declination is 23° 28′ N.: required the time and altitude when its azimuth is the greatest, and also its greatest azimuth.

Ans. Time $9^h \ 47^m \ 53^s$ A. M. (app. time); altitude $=59° \ 11' \ 30''$, greatest azimuth $=$ N. 77° 28′ E.

(135*a*) In latitude 20° N. when the sun's declination is 23° 28′ N.: required the time when the sun will appear stationary in azimuth, the period during which the shadow moved in a contrary direction, and the number of degrees it appeared to go back.

Ans. Time $9^h \ 47^m \ 53^s$: period $4^h \ 24^m \ 15^s$, shadow went back 12° 32′ 30″.

(136*a*) When the sun's declination was 10° 15′ N. and that of the moon was 12° 46′ S., both were observed to rise at the same time: required the latitude of the

place and the time of the observation, the difference of their R. A's. being $1^h\ 53^m\ 42^s$.

Ans. App. time $5^h\ 9^m\ 40^s$ A. M. lat. $50°\ 18'\ 20''$ N.

(137*a*) On what days of the year is the sun on the horizons of Dublin and Pernambuco at the same instant, their respective latitudes being $53°\ 21'$ N. and $8°\ 13'$ S. and their longitudes $6°\ 19'$ W. and $35°\ 5'$ W.

Ans. The four days in the year when the sun's declination is $18°\ 6'$.

(138*a*) At noon on the shortest day, the shadow of a perpendicular stick was 7 times as long as its shadow at noon on the longest day: required the latitude: decl. $23°\ 28'$.

Ans. $\sin. 2l = 2 \sin. 2d$; $l = 38°\ 27'\ 45''$.

(139*a*) Compare the lengths of the shadow of a perpendicular stick at noon in latitude $45°$ N. on the two days when the sun's declination was $15°$ N. and $15°$ S. Ans. Lengths of shadows as 3 : 1.

(140*a*) In latitude $33°\ 30'$ N. and decl. $10°\ 15'$ N. I observed that my shadow bore to my height the proportion of 5 : 3: required the altitude and hour angle of the sun. Ans. Alt. $30°\ 58'$; $h = 3^h\ 58^m\ 4^s$.

(141*a*) The length of the shadow of a perpendicular object was 4 feet, and its longest when sloping was 5 feet: required the sun's altitude. Ans. $36°\ 52'\ 15''$.

(142) The elevation of a cloud was observed to be $20°$, at the same time the sun's altitude was $22°$, the sun and cloud being in the same plane with the observer, and his distance from the shadow 400 yards: required the height of the cloud. Ans. 1468 yards.

(143a) Given the sun's meridian altitude (a) and the hour angle when rising (h) to find the latitude and declination.

Ex. Meridian altitude 56°, hour angle $h = 7^h$.

Ans. Lat. $= 47° \; 23' \; 30''$.

(144) In latitude 45° N. the meridian altitude of the sun $= 30°$: shew that the tangent of quarter the length of the day $= \dfrac{1}{\sqrt[4]{3}}$.

(145a) At a certain place the sun rose at 7^h A. M., and its meridian zenith distance was twice the latitude: required the latitude. Ans. Lat. $= 26° \; 58'$.

(146a) In latitude 45° N., the sun rose at 4^h A. M.: shew that the tangent of the meridian altitude $= 3$.

(147a) In latitude 50° N. when the sun's declination is 5° 38' N.: required the time it will take the body of the sun to rise out of the horizon, its semidiameter being 16'.

Ans. $3^m \; 19^s$.

(148a) Required the time the sun's semidiameter will take to pass the meridian, the declination being 23° 4 and semidiameter 16' 17''.3. Ans. $1^m \; 11^s$.

(149a) A ship in latitude 59° 6' N. and long. 6° 15' E. observed a point of land bearing N. E., and after sailing E. N. E. 6 miles, the point bore N. ½ W.: required the latitude and longitude of the point.

Ans. Lat. 59° 11' 15'', long. 6° 25' 13'' E.

(150) In latitude 50° 48' N. when the sun's declination $= 12° \; 29'$ N. and hour angle $= 2^h \; 53^m \; 1^s$ A. M.: required the azimuth.

Ans. N. 121° 44'. E.

(151*a*) In latitude 45° N., required the difference in the lengths of the longest and shortest days, (decl. 23° 28′.)

Ans. $6^h\ 51^m\ 40^s$.

(152*a*) In what latitude N will the difference between the longest and shortest days be just 6 hours.

Ans. Lat. 46° 46′ N.

(153*a*) At a certain place, when the sun's declination was 10° N., it rose an hour later than when it was 20° N.: required the latitude. Ans. Lat. 52° 27′ N.

(154*a*) In what latitude north will the shortest day be just one third the longest (decl. 23° 29′).

Ans. 58° 27′ N.

(155*a*) And in what latitude will the shortest day be just $\frac{3}{5}$ths the longest. Ans. 41° 24′ N.

(156*a*) At what time in latitudes 50° N. and 60° N., will the sun have the same altitude, its declination being 22° 57′ 15″ N. Ans. $4^h\ 51^m$.

(157*a*) If a ship sail from a certain place *a* miles due east, then *a* miles due south, and then *b* miles due west, and reach the same longitude: required the latitude of the place arrived at.

Ex. $a = 100'$ and $b = 150'$

Ans. 85° 0′ 30″,

(158*a*) Given the apparent and true altitude of a heavenly body, its declination, and observed distance from a terrestial object: to find the true bearing of the object from a given station.

(1) When the object is in the horizon. *Ex.* In latitude 7° 51′ N., the observed altitude of the sun's lower limb was 10° 30′, and observed distance of his nearest limb

from a well defined point of land on the same level with the eye and to the right of the sun was 95° 16′; index correction of the altitude sextant was 0′ 50″—, and that of the other was 1′ 10″ +; the correction for height of the eye (14 feet) in taking the sun's altitude was 3′ 41″ —: required the true bearing of the point of land; the sun's declination being 22° 24′ S., and semidiameter 15′ 45″.

Ans. Bearing N. 19° 0′ 30″ W.

(159*a*) (2) When the object is elevated above the level of the eye, it is necessary to observe its altitude: *Ex.* 2. In the preceding example suppose the object observed to be a mountain the altitude of whose summit is 10°: required the true bearing of that point.

Ans. N. 17° 4′ 0″ W.

The system of rules and operations by which the relative position of any number of points in a tract of country may be determined, so that it may be delineated on a plane surface, is called *Trigonometrical Surveying*. When the extent of country is not great, the subject involves little difficulty, but when a kingdom such as Britain or France is to be surveyed, the aid of astronomy, and other branches of natural philosophy is required. In a survey, the most remarkable objects such as the summit of hills, spires, towers &c, must be chosen as *stations*, and, if necessary, marked by signals. These must be considered as joined by straight lines forming a

chain of triangles, connecting each with all the others. The sides of the triangles should be as long as possible, so as to admit of the stations at any two of the angles being seen from the third: the nearer each triangle approaches to the equilateral form, the better. Supposing a proper disposition of the triangles to have been made, when their angles are known, if a side of any one of them were also known, then the sides of all the others might be found by calculation, and a plan of the country constructed.

A small extent of the earth's surface may be regarded as a plane, and lines perpendicular to it as parallel to one another. In an extensive survey, however, such be that of England, the curvature of the earth must be taken into the account, and then its *figure* and *magnitude* enter as elements into all the calculations. This connection between the figure of the earth and the magnitude and position of lines traced on its surface affords conversely, the means of determining the former when the latter are known: so that such surveys, besides their immediate object, are applicable to the solution of the still more important problem, of finding the magnitude and figure of the earth itself.

When a tract to be surveyed has been covered with a series of triangles, so as to connect the principal points; and all the angles of each, and a side of one are known, the sides of all the triangles may be found by calculation, and a plan made by constructing the triangles on the sides of each other: but in a plan so constructed any error made in the value of one of the sides or angles,

will produce corresponding errors in all the others

To avoid as much as possible this source of error, we may determine the position of a side of one of the triangles with respect to the meridian, by compass, or more accurately by astronomical observations, (see problem 158); and then calculate the distances of all the stations from such meridian, and also the meridional distance of one station from the other: from these distances the position of each station may be laid down in the plan independently of the others, and also the direct distance between any two points may be easily found. The manner of proceeding will appear from the following Problem.

(160). fig. (59) Let A B C D E F be six stations connected by four triangles ABC, BCD, BDE, EDF. the angles are

BAC=79° 20′	CBD=39° 20′
ABC=51 31	BCD=69 28
ACB=49 9	BDC=71 12
DBE=45 28	EDF=62 3
BDE=72 3	DEF=52 25
BED=62 29	DFE=65 32

a side AB of one of the triangles is 4213 yards, and it makes with the meridian NS an angle SAB = 62° 52′ at the point A and the station F makes at the point S an angle ASF with the meridian=52° 40′. It is required to find the points in which perpendiculars from the stations will cut the meridian, and the length of each perpendicular.

Sol. (1). Draw *Bb, Cc, Dd, Ee*, and *Ff* perpendicular to the meridian, and *Bn, Dq*, parallel to it, forming the right angled triangles *ABb, BCm, BDn, EDp, FDq*. Because the angles of the four triangles ABC, BCD, BDE, DEF are given, and also AB, a side of one of them, the five lines AB, BC, BD, DE, DF, may be found from AB and each other: (their logarithms are

AB=3·624591, BD=3·733559, DF=3·578533
BC=3·738255 and DE=3·638690).

(2) In the right angled triangle *ABb*, the side AB (or its log.) and the angle $BAb = 62° 52'$ are known, hence we find, $Ab = 1921·4$ $Bb = 3739·3$ yards.

(3) If from $ABm = 117° 8'$ (the supplement of BAb) the angle ABC = 51° 31′ be taken, there remains $CBm = 65° 37'$; therefore in right angled triangle *CBm*, the angles and the log. of BC are now known, hence we find $Bm = 2259·6$ and $Cm = 4985·2$.

(4) And if from $CBm = 65° 37'$ CBD ($= 39° 20'$) be subtracted, there remains $DBn = 26° 17'$: therefore, in the right angled triangle *DBn* the angles, and the log. of *BD* are known, and hence $Bn = 4854·7$ and $Dn = 2397·6$.

(5.) From $BDq = 153° 43'$ (supp. of *DBn*) subtract $BDE = 72° 3'$, and there remains $EDp = 81° 40'$: then, in the right angled triangle *EDp* the angles, and the log. of *DE* are known, and thence $Dp = 630·7$ $Ep = 4306.1$.

(6) Subtracting $EDF = 62° 3'$ from $EDp = 81° 40'$, there remains $FDq = 19° 37'$, and hence, in the right angled triangle *DFq* we find $Dq = 3569.1$ $Fq = 1272.1$.

To determine the stations, we have now

$Ab \ldots\ldots\ldots = 1921.4$,	$Bb \ldots\ldots\ldots = 3749.3$
$Ac = Ab + Bm = 4181.0$,	$Cc = Cm - Bb = 1235.9$
$Ad = Ab + Bn = 6776.1$,	$Dd = Bb - Dn = 1351.7$
$Ae = Ad + Dp = 7406.8$,	$Ee = Dd + Ep = 1657.8$
$Af = Ad + Dq = 10345{\cdot}2$,	$Ff = Dd + Fq = 2623.8$

By these numbers, the position of each station may be laid down with great accuracy in a plan, independently of the others: also the distance between any two may be readily found · for example,

$$CE = \sqrt{\left\{(Cc + Ee)^2 + (Ae - Ac)^2.\right\}}$$

Lastly. We know by observation that the bearing of F from S is N 52° 40′ E.: hence in the right angled triangle FfS we find $Sf = 2001.2$ yards; adding this to the line Af we have the length of the meridian line between the stations A and $S = 12346.4$ yards.

The above method of measuring the meridian line between two distant stations, furnishes, as we have said, the means of finding the approximate length of a degree, and thence, on the supposition of the earth being an exact sphere, the magnitude of the earth: for we have only to determine by observation the difference of latitude between the stations A and S, and then divide the whole length of the arc, found as above, by the number of degrees contained in it: the result multiplied by 360 will give the circumference of the earth. Thus let us suppose the latitude of A has been found by some of the preceding problems to be 49° 48′ 6″.4 N., and that of S 49° 42′ 4″.86 N., and the length of the line AS to be 12346.4 yards; then by a common proportion we easily

find that the length of a degree is about 69 miles. Picard in 1670 by observations and measurements conducted in a manner similar to the above, found the length of a degree of the meridian in latitude 49° N. to be 121627 yards, which differs only 35 yards from what is now considered as the most exact length: an accuracy however which must be supposed to be quite accidental.

In the Press,—SOLUTIONS of the Problems in this volume.

SOLUTIONS OF PROBLEMS

TRIGONOMETRY PART

DESIGNED AS

AN INTRODUCTION TO NAUTICAL ASTRONOMY.

BY H. W. JEANS,

Royal Naval College, Portsmouth.

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ERRATA IN SOLUTIONS OF PROBLEMS.

PAGE						
23	lines 1, 2, 3, 4,		for	first	read	second
			and for	second	.. ..	first
25	line	16	for	272.7	.. ..	633.4
25	..	18	..	*AED*	.. ..	*AE*
35	..	12	..	*CAD*	.. ..	*CBD*
50	last 3 lines,		for	*tan.* *PZV*, or *tan.* $Z_{,}$	read	*cot.* $Z_{,}$
60	line	2	..	72	.. ..	73
60	..	11	..	38° 12′ 45″	.. ..	30° 12′ 45″
60	..	15	..	*PZM*	.. ..	*PZX*
64	..	17	..	$S_{,}\ P$	.. ..	$S_{,}\ Z$
75	..	17	..	84	.. ..	87
75	..	19	.	85	.. ..	86
78	..	10	..	triangle	.. ..	triangles
		and	..	*XMY*	.. ..	*ZMY*
83	line	2 from end,	for	18′	.. ..	48°
88, 89			for	fig. 94	.. ..	fig. 96
92	line	6 from end,	for	*S*,	.. ..	*S*.
92	lines	2, 3 from end,	for	x and $7x$	.. ..	y and $7y$
94	..	14	for	cloud, *C*	.. ..	cloud *C*,
96	..	13	dele	= after or		
96	..	14	for	*cos.*	.. ..	*cot.*
			and for	$\frac{1}{\sqrt{3}}$	.. ..	$\sqrt{3}$
96	..	15	..	$\frac{h}{2}$	.. ..	$tan\ \frac{h}{2}$.
01	..	13	..	46° 46′	.. ..	41° 24′

103 dele Prob. 156.

March 25th, 1845.

SOLUTIONS OF PROBLEMS IN

PART I.

PROB. 1. (fig. 17).*

Let Cb represent the height of tower = 216 feet.

ab .... width of river.

Aa .... height of spectator's eye = 5 feet,

Suppose AB parallel to horizontal line ab: join CA: then CAB = observed angle of elevation = 47° 56′, and CD = 216 − 5 = 211 feet.

In right ang. tri. ACB, $\frac{AB}{BC}$ = cot. CAB .... (art. 5.*d*).

$$\therefore \text{AB} = \text{BC cot. CAB}.$$

In logarithms,

$$\log \text{AB} = \log \text{BC} + \log \text{cot. CAB} - 10 \ldots (p.\ 30).$$
$$= \log 211 + \log \cot 47^\circ\ 56' - 10.$$

Calculation.

log. 211 .... 2·324282
log. cot. 47° 56′ .. 9·955454

log. AB (− 10) .. 2·279736

$\therefore$ AB ... = ab = 190·4 feet, width of river.

* In fig. 17 mark the line below AB with the letters ab.

PROB. 2. (fig. 18).

Let AC represent the broken piece $= 39 = a$, A the point where it struck the ground, and AB the distance of its top from the base $= 15 = b$. Let BC the part standing $= x$, then, by geometry, $x^2 = a^2 - b^2 = \overline{a+b}.\overline{a-b}$.

In logarithms, $2 \log. x = \log. \overline{a+b} + \log. \overline{a-b}$.

$\therefore \log. x = \frac{1}{2}\left\{\log. (a+b) - \log. (a-b)\right\}$.

Calculation.

$a = 39$	log. 54.... 1·732394
$b = 15$	log. 24.... 1·380211
$a + b = 54$	2)3·112605
$a - b = 24$	log. x ... 1·556302
$=$	$\therefore x$...... $= 36$

$\therefore$ whole height of may-pole $= 39 + 36 = 75$.

PROB. 3. (fig. 19).

Let D and E be the windows; DC $=$ EC the ladder $= 36 = a$; BD height of lower window $= 18{\cdot}9 = b$, and AE height of other window $= 30{\cdot}7 = c$.

Draw FD parallel to AB then FDE is the angle of elevation of E above D.

Let this angle be denoted by θ, BC by x, AC by y.

(1) In triangle ACE.. $y^2 = a^2 - c^2 = \overline{a+c}.\overline{a-c}$.

(2) In triangle CBD.. $x^2 = a^2 - b^2 = \overline{a+b}.\overline{a-b}$.

From these equations, x and y may be found as in the last problem.

(3) To find θ. By art. 5. c. tan. $\theta = \frac{\text{FE}}{\text{FD}} = \frac{c-b}{x+y}$.

Calculation.

(1)

$$
\begin{array}{lll}
a = 36 & \log.\ (a+c) \ldots & 1\cdot824126 \\
c = 30\cdot7 & \log.\ (a-c) \ldots & 0\cdot724276 \\
\hline
a+c = 66\cdot7 & & 2)2\cdot548402 \\
a-c = 5\cdot3 & & \\
\hline
 & \log.\ y \ldots\ldots & 1\cdot274201 \\
 & \therefore\ y \ldots\ldots = & 18\cdot8.
\end{array}
$$

(2)

$$
\begin{array}{lll}
a = 36 & \log.\ (a+b) \ldots & 1\cdot739572 \\
b = 18\cdot9 & \log.\ (a-b) \ldots & 1\cdot232996 \\
\hline
a+b \ldots 54\cdot9 & & 2)2\cdot972568 \\
a-b \ldots 17\cdot1 & & \\
\hline
 & \log.\ x \ldots\ldots & 1\cdot486814 \\
 & \therefore\ x \ldots\ldots = & 30\cdot7
\end{array}
$$

$\because$ breadth of street $= x + y = 49\cdot5$

(3)

To find θ.. tan. $\theta = \frac{c-b}{x+y}$.

$\therefore$ log. tan. $\theta - 10 = \log.\ (c-b) - \log.\ (x+y)$
or log. tan. $\theta = 10 + \log.\ (c-b) - \log.\ (x+y)$.

$$\begin{array}{rll} c = 30{\cdot}7 & \log.\,(c - b) + 10 \;. & 11{\cdot}071882 \\ b = 18{\cdot}9 & \log.\,(x + y) \;\ldots\ldots & 1{\cdot}694605 \\ \hline c - b = 11{\cdot}8 & \log.\tan.\,\theta \;\ldots\ldots\ldots & 9{\cdot}377277 \\ \hline x + y = 49{\cdot}5 & \qquad\qquad \therefore\ \theta\ldots = & 13^\circ\ 24'\ 30''. \end{array}$$

PROB. 4. (fig. 17).

Ex. 1. Let BC represent the height of tower $= x$
AB the base measured........ $= a$
and angle BAC.............. $= d^\circ$

In right angled triangle BAC.. $\frac{x}{a} = \tan.\, d^\circ$

$$\therefore\ x = a \tan.\, d^\circ.$$

In logarithms, $\log. x = \log. a + \log. \tan. d^\circ - 10$

$$\left.\begin{array}{l} a \ = 50 \\ d^\circ = 25^\circ\ 17' \end{array}\right\} \qquad \begin{array}{ll} \log.\, a & = 1{\cdot}698970 \\ \tan.\, d^\circ & = 9{\cdot}674257 \\ \hline \log.\, x \ldots. & 1{\cdot}373227 \end{array}$$

$$\therefore \text{ height } x\ldots = 23{\cdot}62$$

Ex. 2. In this example, since BAC $= 45^\circ$.
$\therefore$ BCA $= 45^\circ$. $\quad \therefore$ BC $=$ BA $= 12\frac{1}{4}$.

PROB. 5. (fig. 20).

Ex. 1. Let AB represent the base $= a = 1000$ yards; C the ship; then $A = 32^\circ\ 10'$; $B = 83^\circ\ 18'$.

In triangle ABC, the side AB and the two angles

A and B are given $\therefore$ C may be found, and thence by Rule III. the side CA.

A.. $= 32° \ 10'$	
B.. $= 83° \ 18'$	By Rule III. $\quad b : c :: \sin. B : \sin. C$
$115 \ 28$	log. c.. $3{\cdot}000000$
180	log sin. B.. $9{\cdot}997024$
$\therefore C = 64 \ 32$	$12{\cdot}997024$
	log sin. C.. $9{\cdot}955609$
	log. b.. $3{\cdot}041415$
	$\therefore \ b = 1100{\cdot}1$ the distance of ship.

Ex. (2).

$A = 80$

$B = 50$

$A + B = 130$

180

$\therefore C = 50$

In this example the angle $C = B$.

$\therefore AC = AB = 2{\cdot}5$ miles.

PROB. 6. (fig. 20).

The solution of this problem is in all respects similar to the last: thus, to find AC, $AC : AB :: \sin. B : \sin. C$

$AB = 2$ miles

$A = 85 \ 10$

$B = 82 \ 45$ $\quad$ To find BC.. $BC : AB :: \sin. A : \sin. C$

$167 \ 55$

180

$\therefore C = 12 \ 5$

whence $\left.\begin{array}{l} AC = 9{\cdot}478 \\ BC = 9{\cdot}52 \end{array}\right\}$ Answer.

PROB. 7. (fig. 21).

Ex. 1. Let AB represent the height of tower, D and C the stations: then DC = 100 yards; ACB = 46° 15′ and ADB = 31° 20′.

(1) In the triangle ADC are given, D = 31° 20′, DC = 100 yards, and DCA = 180° − 46° 15′ : hence (by Rule III) the side AC may be found.

(2) In right angled triangle ACB are given AC and angle ACB, hence AB may be found = 145·9 the height of tower. In 2nd example the height AB = 307·4.

Second Solution.

The height AB may be expressed by one equation in the following manner.

$$\begin{aligned}
\text{Let } x &= \text{height AB}, \ a = \text{base DC} = 100. \\
\alpha &= \text{angle ACB} = 46^\circ\ 15' \\
\beta &= \text{angle ADB} = 32\ \ 20 \\
\therefore\ \alpha - \beta &= \text{angle DAC} = 14\ \ 55
\end{aligned}$$

(1) In triangle ADC $\dfrac{\text{AC}}{a} = \dfrac{\sin.\ \beta}{\sin.\ (\alpha - \beta)}$ $\therefore$ AC =

$\dfrac{a \sin.\ \beta}{\sin.\ (\alpha - \beta)} = a \sin \beta \operatorname{cosec} (\alpha - \beta)$.. art. (5. $a.b$).

(2) In triangle ABC $\dfrac{x}{\text{AC}} = \sin.\ \alpha$ $\therefore$ $x = \text{AC} \sin.\ \alpha$

substituting the value of AC in (1) we have

$$x = a \sin.\ \alpha \sin.\ \beta \operatorname{cosec.} (\alpha - \beta).$$

Calculation.

log. a	2·000000
.. sin. α	9·858756
.. sin. β	9.716017
.. cosec $(\alpha - \beta)$..	0·589368
log. x	2·164141 $\therefore$ $x = 145·9$.

PROB. 8. (fig. 21).

The solution of this problem is similar to the last.

PROB. 9. (fig. 20).

Let B and C be the two ships, A the observer, then dist. of A from C $= 1142 \times a$, of A from B $= 1142 \times b$: hence in the triangle ABC, the two sides AC and AB and the included angle A are given to find the third side BC, (see Rule V.) (If the student have no table of haversines, this problem may be solved by finding in the first place, one of the angles B and C by Rule IV. and then the remaining side by Rule III.)

Calculation of 2nd Ex. by Rule V.

const. log ..	10.602060		10.849485
log. 10	1.000000	sin arc. ..	9.911964
.. 5	0.698970		
hav. 60°	9.397940		0.937521
		log. 1142	3.057666
	2)21.698970		
		log. dist.	3.995187
	10.849485	$\therefore$ distance	$= 9889.9$
	0.698970		
tan. arc.....	10.150515		

PROB. 10. (fig. 21).

In the right angled triangle ABC, are given the side AB the height of mast and the angle BCA (= angle of depression DBC) : hence AC, the distance of the ship C may be found.

PROB. 11. (fig. 22).

Let ABC represent the side of the building, D the place of observer, and O the middle point of the base BC, Draw AO, OD which will be perpendiculars on BC : then $DO = 30$, angle $ADO = 44° \; 1' \; 45''$ and angle $ODC = \frac{1}{2} BDC = 18° \; 26' \; 7''$.

(1) In rt. ang. tri. ADO, $AO = OD$ tan. ADO.

(2) In rt. ang. tri. ODC, $\frac{1}{2} BC = OC = OD$ tan. ODC.

(3) In right angled triangle AOC, $AC = \sqrt{AO^2 + OC^2}$.

PROB. 12. (fig. 23).

Let AB represent the obelisk, C and D the two stations on the declivity.

(1) In triangle ADC ; given DC and all the angles to find AC.

(2) In triangle ACB, given two sides BC, AC and included angle ACB to find the side AB ; the height of obelisk.

PROB. 13. (fig. 21).

(1) In triangle CAB, the angles CAB, ABC and base AB are given to find AC $= 706 \cdot 8$.

(2) In triangle DAB, the angles DAB, DBA and base AB are given to find AD $= 1389 \cdot 4$.

(3) In triangle ADC, the two sides AC, AD and included angle CAD (the difference between CAB and DAB) are given to find CD $= 1174 \cdot 4$ the required distance.

PROB. 14. (fig. 24).

The solution is the same as preceding problem.

PROB. 15. (fig. 24).

Let AB $= x$, AC $= y$, AD $= z$;

Then (1) In triangle CAB. $y : x :: \sin. ABC : \sin. ACB$

$$\therefore\ y = \frac{x. \sin. ABC}{\sin. ACB} = x \sin. ABC \operatorname{cosec} ACB = \cdot 8465x.$$

(2) In triangle ABD, $z : x :: \sin. ABD : \sin. ADB$.

$$\therefore\ z = \frac{x \sin. ABD}{\sin. ADB} = x \sin. ABD \operatorname{cosec} ADB = 1 \cdot 1852x.$$

(3) In triangle CAD we have now

$$\text{CD} = \quad 6594 = a \text{ suppose}$$
$$z = 1 \cdot 1852x = bx\ldots\ldots$$
$$y = \ \cdot 8165x = cx\ldots\ldots$$
$$\text{and angle CAD} = \ 85^\circ\ 46' \ = \text{A}.$$

To find x or AB,

In triangle ACD, $a = \dfrac{\sqrt{4yz} \text{ hav. A}}{\sin. \theta}$ where $\tan. \theta =$

$\dfrac{\sqrt{4yz\text{ hav. A}}}{z-y}$ (see Trig. Part II.): substituting cx and bx, the values of y and z found above we have, $a=$ $\dfrac{x\sqrt{4bc\text{ hav. A}}}{\sin.\ \theta}$ and $\tan.\ \theta = \dfrac{\sqrt{4bc\text{ hav A}}}{b-c}$, from which formulæ x is easily found.

Or thus, (without haversines.)

Find the angles of triangle ACD by Rule (IV), and thence y and x: thus $bx+cx : bx-cx :: \tan \frac{1}{2}(\text{ACD}+\text{ADC}) : \tan. \frac{1}{2}(\text{ACD}-\text{ADC})$, or $b+c : b-c :: \tan \frac{1}{2}(\text{ACD}+\text{ADC}) : \tan \frac{1}{2}(\text{ACD}-\text{ADC})$, whence the angles may be found, and thence y and x.

Prob. 16. (fig. 25).

(1) In triangle DBC, given a, γ, and angle DBC $=\beta-\gamma$ to find BC.

(2) In triangle CBE, given BC, angle BCE $=\beta-a$, and angle CEB $=90^\circ+a$, to find BE, the required height.

Or thus,

$$\text{In tri. DBC} \ldots \frac{\text{BC}}{a}=\frac{\sin \gamma}{\sin.\ (\beta-\gamma)} \quad \therefore \text{BC}=\frac{a \sin \gamma}{\sin.(\beta-\gamma)}$$

$$\text{In tri. BEC} \ldots \frac{\text{BE}}{\text{BC}}=\frac{\sin.\ (\beta-a)}{\sin.\ (90+a)}=\frac{\sin.\ (\beta-a)}{\cos.\ a}.$$

$$\therefore \text{BE}=\frac{\text{BC}.\ \sin.\ (\beta-a)}{\cos.\ a}.$$

Substituting the value of BC found above we have,
$$BE = \frac{a \sin \gamma \sin (\beta - a)}{\sin (\beta - \gamma) \cos a} = a \sin \gamma \sin (\beta - a) \text{ cosec. } (\beta - \gamma) \text{ sec. } a:$$
which equation determines the value of BE

PROB. 17. (fig. 26.)

(1) In triangle ACD, given CD and all the angles to find AD.

(2) In triangle BDE, given DE and all the angles to find DB.

(3) In triangle ADB, given AD, DB and included angle ADB, to find the side AB the required distance.

PROB. 18. (fig. 27).

(1) In triangle ABD, given the three sides to find the angle ABD and thence its supplement ABO.

(2) In triangle ACB, given the three sides to find the angle CAB and its supplement BAO.

(3) In triangle BAO, given BA and the angles to find the sides BO and AO the required distances.

PROB. 19. (fig. 28).

(1) In triangle BAO the side AB and all the angles are known to find AO.

(2) In right angled triangle OAC the angle OAC and AO are known to find AC and OC.

PROB. 20. (fig. 29).

(1) In oblique angled triangle ACB, given side AC = 193 yards, and all the angles to find AB.

(2) In right angled triangle ABH, given angle BAH and hypothenuse to find height BH.

PROB. 21. (fig. 29).

Let BH represent the tower; A the place of the observer due south of tower. AC a straight line drawn due east or at right angles to AH.

(1) In triangle ABH are given, height of tower BH 100 feet and elevation BAH = 50°: to find side AH = 83·9.

(2) In right angled triangle ACH, given AH and AC to find CH = 311·6.

(3) In right angled triangle BCH, given BH and CH to find angle of elevation BCH = 17° 47′ 45″.

PROB. 22. (fig. 30).

Let CD be the perpendicular height of balloon: A and B the two observers: then

angle NAD = 45° 0′ (= N.E.)

NBD = 11° 15′ (= N.*b*.E.);

and CAD = 20° the elevation of balloon.

(1) In horizontal triangle ADB given the side AB and angles NBD and DAB (= suppl. of NAD) to find AD.

(2) In vertical right angled triangle ADC, given AD and angle CAD to find CD.

PROB. 23. (fig. 31).

(1) In triangle BCD given DC, the angle BCD $= 90° - 40°$ and angle BDC $= 90° + 37° 30'$, to find CB.

(2) In right angled triangle ABC given CB and ACB to find height AB, and distance AC of steeple.

PROB. 24. (fig. 32).

(1) In triangle BRS given BR $= 54$, angle RBS $= 90° - 4° 52'$ and angle BSR $= 4° 52' - 4° 2'$, to find RS.

(2) In right angled triangle ARS given RS, and ASR $= 4° 2'$ to find AR the height, and AS the distance.

PROB. 25. (fig. 33).

(1) In triangle ABC given AB $= 500$, and all the angles to find BC.

(2) In right angled triangle DBC given BC, and angle B $= 79° 12'$, to find perpendicular CD the breadth of river.

Or thus,

$$\text{In triangle ABC} \;..\; \frac{BC}{AB} = \frac{\sin. A}{\sin. C} = \frac{\sin. A}{\sin. (A+B)}.$$

$$\therefore \; BC = AB \sin. A \operatorname{cosec.} (A+B).$$

$$\text{In triangle DBC, } DC = BC \sin. B.$$
$$= AB \sin. A \sin. B \operatorname{cosec.} (A+B).$$

PROB. 26. (fig. 34).

Let AB be the sloping base, AD, BF horizontal lines in the vertical plane ACD, and CD the height of the

hill. Then the angles CBF = 46° 15′, CAD = 40° 10′, EBF = EAD = 10° 5′.

∴ CBE = 36° 10′, CAE = 30° 5′, and their difference ACB = 6° 5′.

(1) In triangle ACB find log AC (=3·277228).

(2) In triangle ACD find CD = 1221·2, and adding height of eye 5 feet, the height of hill = 1226·2 feet.

PROB. 27. (fig. 35).

Let A be an object which we may suppose on the top of a hill, and B and C two stations on its sloping side. Conceive a horizontal plane to pass through the lowest C, and let Aa Bb be perpendiculars on that plane, meeting it in a and b: join the points A, B, C in the *oblique* plane and the points a b C in the *horizontal* plane, and draw BD perpendicular to Aa: then AaC BbC will be right angled triangles; and BDab a parallelogram.

There are given BC the distance between the stations; horizontal angle aCb subtended by A and B at C; horizontal angle abC subtended by A and C at B: and the vertical angles ACa BCb the elevations of the object A and upper station B, to find the horizontal distances ba, Ca and the heights Aa, AD.

(1) In the right angled triangle BbC, the hypothenuse BC = 220 and angle BCb = 8° 32′.

∴ Perpendicular Bb = 32·6 and base Cb = 217.6

(2) In the horizontal triangle abC the side $Cb = 217 \cdot 6$ the angles $abC = 70° 15'$, and $aCb = 62° 33'$, $\therefore$ $ab =$ $263 \cdot 1$ and $aC = 279 \cdot 1$.

(3) In the right angled triangle AaC the side $aC =$ $279 \cdot 1$ and the acute angle $ACa = 32° 12'$, $\therefore$ $Aa =$ $175 \cdot 7$.

(4) And since $Da = Bb = 32 \cdot 6$, $\therefore$ $AD =$ $Aa - Bb = 143 \cdot 1$.

PROB. 28. (fig. 35).

(1) In right angled triangle BCb, $BC = 220$ and $BCb = 8° 32'$ $\therefore$ $Bb = Da = 32 \cdot 6$ and $Cb = 217 \cdot 6$

(2) In triangle ABC, the side $BC = 220$ and the three angles are known.

$\therefore$ log. $AC = 2 \cdot 518214$ and $AB = 299 \cdot 5$

(3) In right angled triangle ACa, log. $AC = 2 \cdot 518244$ and angle $ACa = 32° 12'$.

$\therefore$ $Aa = 175 \cdot 7$ and $Ca = 279 \cdot 1$

(4) In right angled triangle ABD, $AD = Aa - Bb$ $= 175 \cdot 7 - 32 \cdot 6 = 143 \cdot 1$; and $AB = 299 \cdot 5$ $\therefore$ $ba =$ $BD = 263 \cdot 1$ yards.

PROB. 29 (fig. 36).

(1) In the triangle CAD, $CD = 549 \cdot 4$ yards, angle $C = 57°$ and $CDA = 14°$ $\therefore$ $CAD = 109°$ and $AD =$ $487 \cdot 27$.

(2) In triangle ABD, $AD = 487 \cdot 27$ and the three angles are known $\therefore$ $AB = 349 \cdot 52$ and $BD = 498 \cdot 7$.

PROB. 30. (fig. 37).

(1) In triangle AFC, given CA = 3000, C = 80° 10′ and CFA = 25° 8′ to find FA = 6959.

(2) In triangle ABF, given FA = 6959, AB = 2000 and included angle FAB = 34° 10′ : hence FB = 5422 the distance required.

PROB. 31. (fig. 38).

Describe a circle passing through the three points A, B, D, and join AE, EB. Then (by Geometry) since angles in the same segment are equal ∴ EAB = EDB and EBA = ADE.

(1) In triangle AEB, the side AB = 5·35 and angle EAB = 15° 30′ and EBA = 12° 15′ ∴ AE = 2·438.

(2) In triangle ABC, the three sides are given to find CAB = 35° 24′ and CAE (= CAB − EAB) = 19° 54′.

(3) In triangle CAE. the sides AC = 2·8, AE = 2.438 and angle CAE = 19° 54′ ∴ angle ACE = 58° 33′.

(4) In triangle ACD, the side AC = 2.8 angle ADC = 12° 15′ and ACD = 58° 33′ ∴ AD = 11·26, CD = 12·46 and angle CAD = 109° 12′

Lastly. In triangle ABD, ADB = 27° 45′, BAD (= CAD − CAB) = 73° 48′ and AB = 5·35 ∴ BD = 11·03.

PROB. 32. (fig. 38).

This problem is similar to the last.

PROB. 33. (fig. 39).

Describe a circle passing through the three points A, B, and D and proceed in a similar manner to problem 31.

PROB. 34. (fig. 40).

Draw BN due north of B, then angle ABN = 8 points or 90°, and CBN (the bearing of C from B) = 4 points, or 45°, ∴ angle ABC = 4 points or 45°, and CAB is a right angle.

In right angled triangle BAC are given AB = 10, and angle BCA = 45°, ∴ AC and BC may be found.

PROB 35. (fig. 41).

In triangle ACB, angle ACB=4 points or 45°, CAB = 4 points or 45°, and side CB = 14·5 are given to find AC and BA.

PROB. 36. (fig. 42).

In triangle ACB the bearing of C from A is N.N.E. and B from A is East. ∴ angle CAB = 6 points also angle ABC = 4 points, and side BC = 10 ∴ AB can be found.

PROB. 37. (fig. 43).

In triangle ACB, the bearing of C from A is N.E.b.N. and of B from A is E.b.N., ∴ angle CAB = 4 points.

The bearing of A from B is W.b.S., and of C from B is W.N.W., $\therefore$ angle ABC = 3 points, and the side AB = 15 miles, hence the other sides may be found

PROB. 38. (fig. 44).

(1) In triangle CAB given AB = 23 miles, CAB = $10\frac{1}{2}$ points, and CBA = $3\frac{1}{2}$ points, $\therefore$ CB = 53·01.

(2) In triangle ABH, AB = 23, ABH = 8 points and HAB = 4 points, $\therefore$ BH = 23.

(3) In triangle CBH, given CB, BH and included angle CBH = $4\frac{1}{2}$ points, to find CH = 42·33 = distance of cape from head-land.

(4) To find bearing of C from H. In triangle CHB, given HB = 23, CH = 42·33, and CBH = $4\frac{1}{2}$ points to find BCH = $2\frac{1}{4}$ points. Hence C bears from H $2\frac{1}{4}$ points to the left; *i. e.*, $2\frac{1}{4}$ points left of W.N.W. or W.$\frac{1}{4}$S.

PROB. 39. (fig. 45).

Let A be the required position of the ship, then in triangle ABC, the bearing of C from B is E.N.E., and A from B is due south. $\therefore$ The angle CBA = 10 points. We have $\therefore$ given BC = 960, AB = 1000 and included angle ABC = 10 points: to find the angle BAC = 20° 32′.

PROB. 40. (fig. 46).

Let A be the required position of a ship, then in triangle ACH, bearing of C from H is W.$\frac{1}{4}$S. and of A from

H is S.b. E. $\frac{1}{2}$ E., $\therefore$ angle CHA $= 9\frac{1}{4}$ points; we have given the two sides CH $= 4{\cdot}23$ and AH $= 2{\cdot}3$ and included angle CHA, to find angle HAC $= 4\frac{1}{2}$ points: hence the bearing of C is $4\frac{1}{2}$ points to the left of H; that is, to the left of N. b. W. $\frac{1}{2}$ W., and the required bearing is $\therefore$ W. N. W.

PROB. 41. (fig. 47).

(1) In horizontal triangle ADS, given AS $= 2640$, SD $= 800$ and angle ASD $= 3$ points $= 33°\ 45'$, to find AD $= 2024{\cdot}2$.

(2) In right angled triangle ABD, AD $= 2024{\cdot}2$ and ADB $= 5°\ 25'$ to find BA the height $= 192$ yards.

PROB. 42. (fig. 48).

In triangle ABC, the angle ACB (the difference of bearings of A and B from C) is $= 3$ points: the bearing of A from B is due west, and of C from B is S.W.b.W. $\therefore$ angle ABC $= 3$ points, and side AC $= 10$ miles, $\therefore$ the side BC may be found.

PROB. 43. (fig. 49).

From D the highest part of muzzle, draw DG parallel to axis AE of piece, then angle BDG $=$ BCA the angle required.

In right angled triangle BDG are given BD $= 77{\cdot}25$ inches, and BG ($=$ BA $-$ DE) $= 6{\cdot}22 - 4{\cdot}92 = 1{\cdot}3$ inches to find angle BDG.

PROB. 44. (fig. 50).

In triangle AED. . . . cot. AED $= \frac{ED}{AD} = \frac{50}{175}$

$\therefore$ AED $= 74^\circ\ 3'\ 15''$ and AEB $= 148^\circ\ 6'\ 30'' =$ angle of the tenaille.

(2) In triangle BAK are given AB $= 350$, BK $= 100$ and included angle ABK $(= 90^\circ - 74^\circ\ 3'\ 15'') = 15^\circ\ 56'\ 45''$: to find AK $= 255{\cdot}33 =$ line of defence.

(3) Since the angle of a regular pentagon $= 108^\circ =$ $HAH_{\prime} + 2$ $BAH_{\prime}$ $\therefore$ $\frac{1}{2} HAH_{\prime} +$ BAH $= 54^\circ$ or $\frac{1}{2}$ $HAH_{\prime} = 54^\circ - 15^\circ\ 56'\ 45'' = 38^\circ\ 3'\ 15''$ $\therefore$ $HAH_{\prime} = 76^\circ\ 6'\ 30''$ the flanked angle.

(4) To find curtain FG. In triangle AED, AE $= \sqrt{AD^2 + DE^2} = \sqrt{175^2 + 50^2} = 182$; hence GE or EF $=$ AG $-$ AE $= 255{\cdot}33 - 182 = 73{\cdot}33$.

Then in isosceles triangle GEF are given two sides GE and EF and included angle GEF $= 148^\circ\ 6'\ 30''$ to find side GF $= 141{\cdot}02$ the curtain.

(5) In triangle FEH are given $EH = 82$, EF $= 73{\cdot}33$ and included angle HEF $= 31^\circ\ 53'\ 30''$ (supplement of $148^\circ\ 6'\ 30''$) to find angle EHF $= 63^\circ$; hence its supplement AHF $= 117^\circ\ 0'$ the angle of the shoulder.

(6) In triangle FEH are given EH and all the angles to find FH $= 43{\cdot}5 =$ length of flank.

(7) And flanked angle GFH $=$ GFE $+$ EFH $= 15^\circ\ 56'\ 45'' + 85^\circ\ 6'\ 30'' = 101^\circ\ 3'\ 15''$.

PROB. 45*a*. (fig. 51).

Let AE represent the part standing the first time.
AC .. .. second ..
EF .. .. .. broken .. first ..
and CD .. .. second ..
then $AD = 20$ and $AF = 30$
let $AE = x$ $\therefore$ $AC = x + 5$
$CD = y$ $\therefore$ $EF = y + 5$
and $x + y + 5 =$ height of may-pole.

In triangle FAE.... $x^2 + 30^2 = (y + 5)^2$ .. (1).
In triangle ACD.... $(x + 5)^2 + 20^2 = y^2$ .. (2).

$$\text{By (1)} \quad x^2 + 900 = y^2 + 10y + 25.$$
$$\text{By (2)} \quad x^2 + 10x + 425 = y^2$$
$$\text{subtracting (1) from (2)} \quad 10x - 475 = -10y - 25$$
$$\text{or } 10(x + y) = 450$$
$$\therefore x + y = 45$$
$$\text{and } x + y + 5 = 50 \text{ the height.}$$

PROB. 46*a*. (fig. 52).

Let $x =$ BD the horizontal distance of A from B
$y =$ CD C
$z =$ AD the height of hill
and $\theta =$ angle of elevation of A from C = ACD
$a =$ elevation of A at B $= 20^\circ$
$b =$ horizontal angle BCD or the bearing of A from C $= 67^\circ\ 30'$.

In right angled horizontal triangle BCD $x = y$ sin. b (1)
In vertical triangle ADB . . . . $z = x$ tan. a (2)
. . . . ADC. . . . . $z = y$ tan. θ (3)

From equations (2) and (3) x tan. $a = y$ tan. θ substituting the value of x from (1) we have

$$y \text{ sin. } b \text{ tan. } a = y \text{ tan. } \theta$$

$$\therefore \text{ tan. } \theta = \text{ sin. } b \text{ tan. } a$$

log. sin. b.	9·965615
tan. a	9·561066
tan θ	9·526681 $\therefore$ $\theta = 18° \ 36'$

PROB. 47a, (fig. 53).

(1) In the triangle ABC, find one of the angles at the base as B, and thence in the right angled triangle ADB the perpendicular AD.

or thus: (without using tables).

Let AD$= x$; and segment BD $= y$; a, b, c the three sides of the triangle; $a = 40$, $b = 30$, $c = 20$.

In right angled triangle ABD. . . . $x^2 + y^2 = c^2$. . . . (1)
. . . . ADC . . $x^2 + (a - y)^2 = b^2$. . (2)

Subtract (1) from (2) we have $(a - y)^2 - y^2 = b^2 - c^2$

$$\text{whence } y = \frac{a^2 + c^2 - b2}{2a} = 13·75$$

and $x = \sqrt{c^2 - y^2} = 14·523$, the perpendicular required.

PROB. 48. (fig. 54).

Let DAC $= a = 23° \ 50' \ 15''$; CAB $= b = 93° \ 4' \ 20''$ ABC $= c = 54° \ 28' \ 30''$ and side AB $= 416$.

(1) In triangle ABC $\frac{AC}{AB} = \frac{\sin. c}{\sin. ACB} = \frac{\sin. c}{\sin (b+c)}$

or, $AC = \sin. c \operatorname{cosec.} (b + c)\, AB$.

(2) In triangle DAC.. $DC = AC \tan. a$

$= \sin. c \operatorname{cosec} (b + c) \tan. a\, AB$.

PROB. 49*a*. (fig. 52).

Let distance BD of first station $= x$

.... CD .. second .... $= y$

height AD .. tower .. . $= z$

angle $ABD = a = 23° 18'$, angle $ACD = \beta = 21° 16'$ and side $BC = 300$.

In triangle ABD .... $z = x \tan. a$ ...... (1).

In triangle ACD .... $z = y \tan. \beta$ ..... (2).

$\therefore x \tan a = y \tan. \beta$ and $\frac{x}{y} = \frac{\tan \beta}{\tan a} = \cos. BDC$.

whence $BDC = 25° 20' 45''$.

Then in triangle BDC .. $BD = BC \cot. BDC =$

PROB. 50*a*. (fig. 55).

Let AED, height of man $= a = 6$ feet

AB height of column $= a + b = 200$.

and BC height of statue $= c = 50$

Let $\theta = ADE$ or BDC the angle subtended at D the place of spectator by a or c.

$x =$ BDE and $y =$ AD the width of river.

(1) In triangle AED.. $\frac{a}{y} = \tan. \theta$

(2) In triangle ABD $\therefore \frac{a+b}{y} = \tan.\ (\theta+x)$

(3) .... ACD.. $\frac{a+b+c}{y} = \tan.\ (2\theta+x) =$

$$\tan.\ (\overline{\theta+x}+\theta) = \frac{\tan.\ (\theta+x)+\tan.\ \theta}{1-\tan.\ (\theta+x)\tan.\ \theta} = \frac{\frac{a+b}{y}+\frac{a}{y}}{1-\frac{a+b}{y}\cdot\frac{a}{y}}$$

$$\left(\text{from (1) and (2)}\right) = \frac{(2a+b)y}{y^2-(a+b)a}.$$

$$\therefore \frac{a+b+c}{y^2} = \frac{2a+b}{y^2-(a+b)a}$$

Substituting in this equation the values of a, b, c, we have

$$\frac{250}{y^2} = \frac{206}{y^2-1200},$$

or $\frac{206}{250} = \frac{y^2-1200}{y^2} = 1 - \frac{1200}{y^2}$ wherefore,

$\frac{1200}{y^2} = 1 - \frac{206}{250} = \frac{22}{125}$ and $y = 82{\cdot}5$.

PROB. 51*a*. (fig. 56).

Let BC height of tower $= x$; CD the flagstaff $= 4$ yards $= a$; AB distance of observer $= 100$ yards $= b$; $\alpha =$ DAC $= 48'\ 20''$, and $\theta =$ angle BAC subtended by tower.

(1) In triangle BAC .. $\frac{x}{b} = \tan.\ \theta$;

in triangle BAD $\frac{x+a}{b} = \tan.\ (\theta + \alpha) =$

$$\frac{\tan.\ \theta + \tan.\ \alpha}{1-\tan.\ \theta\ \tan.\ \alpha} = \frac{x + b\tan.\ \alpha}{b - x\tan.\ \alpha} \text{.. from (1).}$$

By solving the quadratic equation $\frac{x+a}{b} = \frac{x+b \tan. a}{b-x \tan. a}$

we have.. $x = -\frac{a}{2} \pm \sqrt{\left\{\frac{a^2}{4} + \frac{ab-b^2 \tan. a}{\tan. a}\right\}}$

Calculation.

log. tan. a = tabular log. tan. $a - 10$.. (Trigonometry, Pt. I. Art. 32) $= 8{\cdot}117959 - 10 = \bar{2}{\cdot}147959$.

its Nat. No. or natural tang. $a = {\cdot}01406$.

$$\therefore x = -2 \pm \sqrt{\left\{4 + \frac{400 - 10000 \times {\cdot}01406}{{\cdot}01406}\right\}}$$

$= 133{\cdot}8$ yards $= 401{\cdot}4$ feet (See also Calculation of Prob. 69*a*.)

Or thus,

In problems of this kind, where one of the angles concerned is only a few minutes, the *circular* measure of the angle may sometimes be used with advantage.

Let height of flag-staff CD $= a$, distance of spectator AB $= b$, height of tower BC $= x$, angle BAC $= a$.

In AD take AE = AC and join CE.

Since the angle DAC is small the line CE may be supposed to be perpendicular to AE and AC; and circular measure of angle DAC or $\theta = \frac{\text{arc}}{\text{rad}} = \tan. \theta$ nearly

$= \frac{\text{EC}}{\text{AC}} = \frac{a \cos. \text{ECD}}{b \sec. a} = \frac{a}{b} \cos.^2 a$ (since ECD $= a$)

$\therefore\ \cos.^2 a = \frac{b\theta}{a}$; but $\theta = \frac{\text{arc}}{\text{rad}} = \frac{48'\ 20''}{57^\circ.29577} = \frac{0 \cdot 80555}{57.29577}$;

and $b = 100$, $a = 4$, $\therefore\ \cos.^2 a = \frac{25 \times 80555}{5729577}$

$\therefore\ a = 53^\circ\ 38'\ 30''$ and $x = b$ tan. $a = 407 \cdot 5$ feet.

Prob. 52.

Since the area of a parallelogram is equal to the product of its base by its altitude, and the area of triangle $= \frac{1}{2}$ area of a parallelogram with same base and alt. (Hind's Alg.) $\therefore$ area triangle $= \frac{1}{2}$ base $\times$ alt. $= \frac{1}{2} \times 40 \times 30 = 600$ square feet.

Prob. 53*a*.

Let $x =$ base of required triangle and $y =$ perpendicular then $xy = 48$ and $x : y :: 12 : 16 :: 3 : 4$ or $4x = 3y$, whence $x = \frac{48}{y}$ and $x = \frac{3y}{4}$ $\therefore\ \frac{48}{y} = \frac{3y}{4}$ and $\therefore\ y = 8$, $x = 6$ and hypothenuse $= 10$.

Prob. 54*a*.

Let $x =$ side of triangle ABC. Now area $=$

$\left\{ \sqrt{S.(S-a).(S-b).(S-c)} \right\}$ where $S = \frac{a+b+c}{2}$

$= \frac{3x}{2}$ (since $a = b = c = x$).

$\therefore$ Area $= \sqrt{\left\{ \frac{3x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \cdot \frac{x}{2} \right\}} = \frac{\sqrt{3}}{4} x^2 = 180$ (by the question) $\therefore\ x = 20 \cdot 389$.

Prob. 55*a*. (fig. 53).

Find one of the sides AC, (Rule III) and thence the area, having two sides BC, AC and included angle C given (Rule VI).

Or thus, in ABD .... $\dfrac{AB}{BC} = \dfrac{\sin. C}{\sin. (B + C)}$

$$\therefore AB = \frac{BC \sin. C}{\sin. (B + C)}$$

and 2 area $= AB.BC \sin. B = \dfrac{BC^2 \sin. B \sin. C}{\sin. (B + C)}$

Prob. 56*a*. (fig. 53).

In triangle ABC, let $a = 5$, $b = 3$ and $x =$ third side, then area$^2 = 36 = S.(S-a).(S-b).(S-x)$

where $S = \dfrac{a + b + x}{2} = \dfrac{8 + x}{2}$.

$$\therefore \; 36 = \left(\frac{8+x}{2}\right).\left(\frac{8+x}{2} - 5\right).\left(\frac{8+x}{2} - 3\right).\left(\frac{8+x}{2} - x\right).$$

$$= \frac{(8+x)\,(x-2).(2+x).(8-x)}{16} = \frac{(64-x^2).(x^2-4)}{16}.$$

whence, solving this quadratic $x = 4$.

Prob. 57*a*, (fig. 57).

Let AD alt. of tri. ABC $= a = 10$; $x =$ base BC.
$y =$ base of triangle AEF, and $z =$ its altitude.

area tri. AEF $= \frac{1}{2}$ area ABC or $\dfrac{yz}{2} = \dfrac{1}{2}, \dfrac{ax}{2}$.

$\therefore\ yz = \frac{1}{2}\ ax\,..\,(1)$; but by similar tri. .. $\frac{z}{y} = \frac{a}{x}$,

or $y = \frac{zx}{a}$: substituting the value of y from (1)

we have $\frac{z^2x}{a} = \frac{ax}{2}\ \therefore\ z = \frac{a}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7{\cdot}071$.

$\therefore$ dist. from base $= 10 - 7{\cdot}071 = 2{\cdot}929$.

PROB. 58*a*.

Let $x =$ the less side of the rectangle and $y =$ the greater side : then $2x =$ greater side of triangle, and $y =$ its hypothenuse. $\therefore\ \sqrt{y^2 - 4x^2} =$ third side of triangle.
also $2x =$ side of square whose area $=$ that of the two allotments.

$$\therefore\ 4x^2 = xy + x\sqrt{y^2 - 4x^2}\,........\,(1).$$

Length of paling for rectangle and triangle $=$
$$2\,(x+y) + 2\,x + y + \sqrt{y^2 - 4x^2},$$
and length of paling for square $= 8x$.

$$\therefore\ 8x + 55 = 2\,(x+y) + 2x + y + \sqrt{y^2 - 4x^2}\,..\,(2)$$

From equation (1) we find $y = \frac{5x}{2}$ which substituted

in (2) gives $x = 11\ \therefore\ y = 27{\cdot}5$,

hence, area rectangle $= xy = 302{\cdot}5$,

and area triangle $= x\sqrt{y^2 - 4x^2} = 181{\cdot}5$.

Prob. 59*a*.

Let $x + y$, x, and $x - y$ represent the three sides of the right angled triangle.

then $(x + y)^2 = x^2 + (x-y)^2$.......... (1)

and $\frac{1}{2}(x - y) \cdot x = 216$.............. (2)

From these equations the values of x and y are found, and thence the sides of triangle.

Prob. 60*a*.

Let $x =$ one side of equilateral triangle.

then area $= \frac{1}{2}x^2 \sin 60° = \frac{1}{2}x^2 . \frac{1}{2}\sqrt{3} = \frac{x^2}{4}\sqrt{3}$

$\therefore$ Expense of paving $= 8 \cdot \frac{x^2}{4} \cdot \sqrt{3} = 2x^2\sqrt{3}$

and expense of pallisading three sides $= 3x \cdot 84 = 252x$.

$\therefore 2x^2\sqrt{3} = 252x$. and $x = 72 \cdot 74$.

Prob. 61*a*.

Let AB one side of a regular polygon $= a$, number sides $= n$: let C be the center of polygon.

Draw CD perpendicular to AB: join CA, CB. then

angle $ACB = \frac{360°}{n} \therefore ACD = \frac{360}{2n} = \frac{180}{n}$.

Now area polygon $= n \times$ Area triangle $ACB =$

$2n$. Area ACD $= 2n \cdot \frac{1}{2}$ AD.DC $= n$. AD.DC.

But DC = AD cot. ACD = AD cot. $\frac{180^\circ}{n}$

$$\therefore \text{ Area polygon } = n.\text{AD}^2 \text{ cot. } \frac{180^\circ}{n}$$

and AD $= \frac{a}{2}$. $\therefore$ Area polygon $= \frac{na^2}{4}$ cot. $\frac{180^\circ}{n}$.

Prob. 62*a*.

In last problem : Area A$= \frac{na^2}{4}$ cot. $\frac{180^\circ}{n}$.

$$\therefore a^2 = \frac{4\text{A}}{n \text{ cot.} \frac{180}{n}} = \frac{4\text{A}}{n} \text{ tan.} \frac{180^\circ}{n}$$

$$\therefore a = \sqrt{\frac{4\text{A}}{n} \text{ tan. } \frac{180^\circ}{n}} = \sqrt{\frac{4 \times 1236 \cdot 1}{8} \text{ tan. } 22\tfrac{1}{2}^\circ}.$$

Calculation of Ex. 1.

log. 4............	0·602060
.. 1236·1	3·092043
.. tan. $22\frac{1}{2}^\circ$......	9·617224
ar. co. log. 8......	9·096910
	2)2·408237
log. *a*............	1·204118 $\therefore$ a=16.

PROB. 63*a*. (fig. 58).

The area of triangle $ABC = \frac{1}{2} AB . AC \sin. BAC$ $= \frac{1}{2}n\, AD . AC \sin. BAC$.. (1).. since $n.AD = AB$.

$$\text{In triangle } AED.. \frac{AE}{AD} = \frac{\sin. ADE}{\sin. AED} = \frac{\sin. BAC}{\sin. CAG}.$$

$$\therefore AE \sin. CAG = AD \sin. BAC.$$

Substituting this value of $AD.\sin. BAC$ in (1).

$$\text{Area triangle } ABC = \tfrac{1}{2}n\, AC \cdot AE \sin. CAG:$$

but $AC.AE = AF^2$, since AF is taken a mean proportional to AC and AE.

$$\therefore \text{Area triangle } ABC = \tfrac{1}{2}\, n . AF^2 \sin. CAG$$
$$= n \text{ triangle } AFH = \text{Area polygon.}$$

PROB. 64. (fig. 60).

Let CB represent length of ship = 160 feet. A the place of observer: then $CAB = 20' \, 15''$, also the angles ABC, ACB are nearly equal to each other since the side CB (subtended by an angle of a few minutes) is small compared with AB or AC.

$\therefore$ each of the angles SBC or SCE may be considered to be the course $= 22^\circ \, 30'$.

$$\text{In triangle } ABC.. \frac{AB}{BC} = \frac{\sin. C}{\sin. A} = \frac{\sin. 22^\circ \, 30'}{\sin. 20' \, 15''}.$$

$$\therefore AB = BC \sin. 22^\circ \, 30' \text{ cosec. } 20' \, 15'' = 10247 \text{ ft.}$$

Otherwise,

Let AD = AB, join DB, then (the angle A being small) each of the angles ABD, ADB is nearly equal to

a right angle: and DB may be considered as the arc subtending the angle A. Let circular measure of the

$$\text{angle A} = \theta = \frac{\text{arc}}{\text{rad.}} = \frac{\text{DB}}{\text{AB}} = \frac{20'\ 15''}{57^\circ \cdot 29577}$$

$$\therefore \text{ AB} = \text{DB}\frac{57^\circ \cdot 29577}{20'\ 15''} = \frac{\text{BC sin. } 22^\circ\ 30' \times 57^\circ \cdot 29577}{20'\ 15''}$$

since $\frac{\text{BD}}{\text{BC}}$ = sin. BCD nearly, Reducing to seconds and dividing by 1760×3 to obtain the distance in miles, we have

$$\text{AB} = \frac{160 \times \text{sin. } 22^\circ\ 30' \times 57 \cdot 29577 \times 60 \times 60}{1215 \times 1760 \times 3} = 1 \cdot 94$$

miles.

PROB. 65*a*. (fig. 61).

Let AB represent the meridian; angle BAC $= x^\circ$ and angle ABC $= x^\circ + 10^\circ$: then C is the point where the ships meet; and the perpendicular CD the distance from the meridian = 100 miles; also since the sides AC and CB are to each other as 3 : 2, let AC $= 3y$ and CB $= 2y$.

Then $\frac{3y + 2y}{3y - 2y} = \frac{\text{tan.} \frac{1}{2}(x + 10 + x)}{\text{tan.} \frac{1}{2}(x + 10 - x)}$ (Rule IV. Note).

or $5 = \frac{\text{tan. } (x + 5)}{\text{tan. } 5}$ $\therefore$ tan. $(x + 5) = 5$ tan. 5°

whence $x + 5 = 23^\circ\ 37'\ 45''$ and $x = 18^\circ\ 37'\ 45''$ and difference of lat. AB = AD + DB = CD cot. x + CD cot. $(x + 10) = 296 \cdot 6 + 183 \cdot 2 = 479 \cdot 8$.

PROB. 66*a*. (fig. 62).

Let AB represent the height 6 feet $=$ 2 yards : AA′ the diameter of the earth : draw the tangent BD : then BD will be the distance seen by spectator (neglecting the effects of refraction, &c.)

By Geometry.... $BD^2 = BA.BA' = 2 \times 14080002$ (if we suppose diameter$=$8000 miles$=$14080000 yards)

$$\therefore BD = 5307.$$

PROB. 67*a*. (fig. 62).

Let BH be drawn perpendicular to BA. then the angle HBD (the angle of depression of D a point in the horizon) $= 90° - CAD = DCA$; since angle at D $= 90°$ $\therefore$ angle DCA $= 2° \, 13' \, 27''$.

Let a = BA the height of mountain = 3 miles
x = CA or CD the radius of the earth

$$\text{then sec. } C = \frac{CB}{CD} = \frac{x+a}{x} = 1 + \frac{a}{x}$$

$$\therefore \frac{a}{x} = \text{sec. } C - 1 = \frac{1}{\cos. C} - 1 = \frac{1 - \cos. C}{\cos. C} = \frac{2 \sin.^2 \frac{C}{2}}{\cos. C}$$

$$\therefore 2x = a \cos. C \operatorname{cosec.}^2 \frac{C}{2}.$$

Calculation.

$$\log. 2x = \log. 3 + \log. \cos. C + 2 \log. \operatorname{cosec} \frac{C}{2} - 30.$$

$$\therefore 2x = 7952 \text{ miles.}$$

PROB. 68*a*. (fig. 62).

Let $a =$ DC or AC the radius of earth.
$x =$ BA height of mountain.

$$\text{then} \frac{x+a}{a} = \sec. C, \text{ or } \frac{x}{a} = \sec. C - 1 = \frac{1-\cos C}{\cos. C}$$

$$= \frac{2 \sin.^2 \frac{C}{2}}{\cos. C} \quad \therefore\ x = 2a \sin.^2 \frac{C}{2} \sec. C = 1{\cdot}402 \text{ miles.}$$

PROB. 69*a*. (fig. 63).

This problem may be solved geometrically as follows: Upon the line AB = 400 yards, the distance between the objects A and B describe a segment of a circle containing an angle equal to the given angle, (35° 10′); (Euclid III. 33) and measure a perpendicular distance CD = 560 yards, the width of the river: through D draw EDF parallel to base AB, cutting the circle in E and F: then either of the points E, F will correspond to the station on the other side of the river: and the lines EA, EB or FA, FB measured on the same scale as AB will be the required distances.

Second solution (Trigonometrically).

(1) In right angled triangle AGC, given AC = 200, and angle AGC = AEB = 35° 10′ to find radius AG = 347.75 and GC = 283.87. and angle CAG = 54°50′.

(2) ∴ GD = CD − GC = 560 − 283.87 = 276.13.

(3) In right angled triangle EDG are given GD = 276.13 radius EG = 347.75 to find angle EGD = 37° 19′ 30″.

(4) Then the angle AGE $= 180° - (AGC+EGD) =$ $180 - (35° \; 10' + 37° \; 19' \; 30'') = 107° \; 30' \; 30''$.

(5) Draw GH perpendicular to AE. We have now given AG and angle AGH$=53° \; 45' \; 15''$ in right angled triangle AGH to find AH $= \frac{1}{2}$ AE whence AE$=560.1$.

(6) To find the other side EB, there are given AE and AB and included angle EAB $=$ CAG $+$ GAH $=$ $54° \; 50' + 36° \; 14' \; 45'' = 91° \; 4' \; 45''$ to find EB$=694.37$.

Or thus (analytically)

By the construction of fig. the two objects are on the same side of the station. Let A be the nearest object to station, B the more distant.

AB, distance of objects $= a = 400$; produce BA to K meeting perpendicular from station E, and let AK $= x$, AE $= y$, BE $= z$, angle AEK $= \theta$, angle AEB between objects $= \alpha = 35° \; 10'$, and perpendicular EK $=$ $b = 560$.

In triangle AEK .. $\frac{x}{b} = \tan \theta \ldots\ldots\ldots\ldots\ldots$ (1)

$\ldots\ldots\ldots$ BEK .. $\frac{a+x}{b} = \tan(\theta + \alpha) = \frac{\tan.\alpha + \tan\theta}{1 - \tan.\alpha \; \tan.\theta}$

$$\therefore \; a+x = \frac{b \tan.\alpha + b \tan.\theta}{1 - \tan.\alpha \tan.\theta}$$

$$(\text{from } 1) = \frac{b \tan.\alpha + x}{1 - \tan.\alpha \frac{x}{b}} = \frac{b^2 \tan.\alpha + bx}{b - x \tan.\alpha}$$

$$\therefore \; x^2 + ax = \frac{ab - b^2 \tan.\alpha}{\tan.\alpha}$$

and solving quadratic, $x = -\frac{a}{2} \pm \sqrt{\left\{ \frac{a^2}{4} + \frac{ab - b^2 \tan.\alpha}{\tan.\alpha} \right\}}$

To find numerical value of this expression,
First find b^2 tan. a.

log. b......	2.748188	log.. 3053	3.484727
log. b......	2.748188	cot. a......	0.152087
tan. a......	9.817913		3.636814
	5.314289		

$\therefore b^2$ tan. a.. 220947

ab 224000

$\therefore ab - b^2$ tan. $a = 3053$

$$\therefore \frac{ab - b^2 \text{ tan. } a}{\text{tan. } a} \ldots = 4333$$

$$\frac{a^2}{4} \ldots\ldots\ldots = 40000$$

$$\therefore \frac{a^2}{4} + \frac{ab - b^2 \text{ tan. } a}{\text{tan. } a} \ldots = 44333$$

and $\sqrt{44333} = 210.5 \;\therefore x = -200 \pm 210.5 = 10.5$.
With this value of x we find $y = 560.1$ and $z = 694.3$.

PROB. 70. (fig. 64).

Let AB represent the tower, CD the road, C the place of spectator, then angle ACB $= 20°$ the elevation of tower at C, and ACD $= 30°$ the angle between line joining the top and road CD. Let BD be the nearest distance of tower from road ($=$ 200 feet): then plane of triangle ABD is perpendicular to road and ADC is a right angle.

In triangle ABC .. AB $=$ AC sin. ACB $=$ AC sin. 20°
In triangle ADC .. AD $=$ AC sin. ACD $=$ AC sin. 30

$$\therefore \frac{AB}{AD} = \frac{\text{sin. } 20°}{\text{sin. } 30°} \text{ but } \frac{AB}{AD} = \text{sin. ADB}$$

$$\therefore \text{sin. ADB} = \frac{\text{sin. } 20}{\text{sin. } 30} \text{ or ADB} = 43° \; 9' \; 15''$$

then in triangle ABD .... AB $=$ DB tan. ADB $=$ 200.tan. $43° \; 9' \; 15'' = 187.5$.

Or thus ;

The angle at C is a solid angle formed by the vertical angle ACB, the oblique angle ACD, and horizontal angle BCD which call θ: let ACB $= a$, ACD $= \beta$. To find the horizontal angle θ we may consider the point C the center of a sphere, and the three angles a, β, θ, may then be represented by the three sides of a right angled spherical triangle; of which a and θ contain the right angle and β is the hypothenuse.

By Rule XV, $\cos. \beta = \cos. a \cos. \theta$ whence $\cos. \theta = \cos. \beta \sec. a$, which determines $\theta = 22° 50' 15''$: then in right angled triangle BDC are given BD and angle BCD to find CB, which with the angle ACB will give AB, the height of tower.

PROB. 71*a*.

Let $7x$ and $4x$ represent the sides, and included angle $= 129° 34'$: let θ and ϕ denote the other two angles.

$$\therefore \theta + \phi = 180° - 129° 34' = 50° 26'.$$

$$\text{Now } \frac{7x + 4x}{7x - 4x} = \frac{\tan. \frac{1}{2}(\theta + \phi)}{\tan. \frac{1}{2}(\theta - \phi)} \ldots\ldots \text{(Rule IV. note).}$$

$$\text{or } \frac{11}{3} = \frac{\tan. 25° 13'}{\tan. \frac{1}{2}(\theta - \phi)} \therefore \tan. \tfrac{1}{2}(\theta - \phi) = \frac{3}{11} \tan. 25° 13'$$

which determines $\theta - \phi$ and $\therefore$ with $\theta + \phi$ already known, the angles θ and ϕ may be found.

To find numerical value of this expression,

First find b^2 tan. a.

log. b......	2.748188	log.. 3053	3.484727
log. b......	2.748188	cot. a......	0.152087
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	5.344289		

$\therefore b^2$ tan. a.. 220947

ab 224000

$\therefore ab - b^2 \tan. a = 3053$

$$\therefore \frac{ab - b^2 \tan. a}{\tan. a} .. = 4333$$

$$\frac{a^2}{4} .... = 40000$$

$$\therefore \frac{a^2}{4} + \frac{ab - b^2 \tan. a}{\tan. a} .. = 44333$$

and $\sqrt{44333} = 210.5 \ \therefore x = -200 + 210.5 = 10.5$.
With this value of x we find $y = 560.1$ and $z = 694.3$.

PROB. 70. (fig. 64).

Let AB represent the tower, CD the road, C the place of spectator, then angle ACB $= 20°$ the elevation of tower at C, and ACD $= 30°$ the angle between line joining the top and road CD. Let BD be the nearest distance of tower from road ($= 200$ feet): then plane of triangle ABD is perpendicular to road and ADC is a right angle.

In triangle ABC .. AB $=$ AC sin. ACB $=$ AC sin. $20°$
In triangle ADC .. AD $=$ AC sin. ACD $=$ AC sin. 30

$$\therefore \frac{AB}{AD} = \frac{\sin. 20°}{\sin. 30°} \text{ but } \frac{AB}{AD} = \sin. ADB$$

$$\therefore \sin. ADB = \frac{\sin. 20}{\sin. 30} \text{ or } ADB = 43° \ 9' \ 15''$$

then in triangle ABD .... AB $=$ DB tan. ADB $=$
200.tan. $43° \ 9' \ 15'' = 187.5$.

Or thus ;

The angle at C is a solid angle formed by the vertical angle ACB, the oblique angle ACD, and horizontal angle BCD which call θ: let ACB $= a$, ACD $= \beta$. To find the horizontal angle θ we may consider the point C the center of a sphere, and the three angles a, β, θ, may then be represented by the three sides of a right angled spherical triangle; of which a and θ contain the right angle and β is the hypothenuse.

By Rule XV, $\cos. \beta = \cos. a \cos. \theta$ whence $\cos. \theta = \cos. \beta \sec. a$, which determines $\theta = 22° 50' 15''$: then in right angled triangle BDC are given BD and angle BCD to find CB, which with the angle ACB will give AB, the height of tower.

Prob. 71*a*.

Let $7x$ and $4x$ represent the sides, and included angle $= 129° 34'$: let θ and ϕ denote the other two angles.

$$\therefore\ \theta + \phi = 180° - 129° 34' = 50° 26'$$

$$\text{Now } \frac{7x+4x}{7x-4x} = \frac{\tan. \frac{1}{2}(\theta+\phi)}{\tan. \frac{1}{2}(\theta-\phi)} \ldots\ldots \text{(Rule IV. note).}$$

$$\text{or } \frac{11}{3} = \frac{\tan. 25° 13'}{\tan. \frac{1}{2}(\theta-\phi)} \therefore \tan. \tfrac{1}{2}(\theta-\phi) = \frac{3}{11} \tan. 25° 13'$$

which determines $\theta - \phi$ and $\therefore$ with $\theta + \phi$ already known, the angles θ and ϕ may be found.

PROB. 72*a*.

Let θ, 2θ, 3θ denote the angles.
x, y, z the sides opposite to them respectively.
then, $x+y+z=6 \ldots\ldots (1)$

$$\theta+2\theta+3\theta=180^\circ \quad \therefore \theta=30^\circ.$$

Now $\frac{x}{z}=\frac{\sin.\theta}{\sin.3\theta}=\frac{\sin.30}{\sin.90}=\frac{1}{2}. \quad \therefore 2x=z \ldots (2)$

and $\frac{x}{y}=\frac{\sin.\theta}{\sin.2\theta}=\frac{\sin.30}{\sin.60}=\frac{\frac{1}{2}}{\frac{1}{2}\sqrt{3}}=\frac{1}{\sqrt{3}}$

$$\therefore x\sqrt{3}=y \ldots (3).$$

Substitute in (1) the values of y and z in (3) and (2), we have, $x+x\sqrt{3}+2x=6$ or $x=\frac{6}{3+\sqrt{3}}=3-\sqrt{3}=1{\cdot}268.$

$$\therefore z=2x=2{\cdot}536 \text{ and } y=x\sqrt{3}=2{\cdot}196.$$

PROB. 73*a*.

Let θ, 2θ, 4θ denote the angles.
x, y, z the sides opposite to them respectively.
then $x+y+z=100 \ldots\ldots\ldots (1)$
and $\theta+2\theta+4\theta=180^\circ \quad \therefore \theta=25^\circ\, 42'\, 51''.$

Now $\frac{z}{y}=\frac{\sin.4\theta}{\sin.2\theta}=\frac{2\sin.2\theta\cos.2\theta}{\sin.2\theta}=2\cos.2\theta=1{\cdot}247$

$$\therefore z=1{\cdot}247y.$$

$\frac{y}{x}=\frac{\sin.2\theta}{\sin.\theta}=\frac{2\sin.\theta\cos.\theta}{\sin.\theta}=2\cos.\theta=1{\cdot}802 \therefore x=\frac{y}{1{\cdot}802}$

substituting in (1); $\frac{y}{1{\cdot}802}+y+1{\cdot}247y=100 \therefore y=35{\cdot}69$

whence $x=19{\cdot}8$ and $z=44{\cdot}51$.

See Prob. 85*a* for another solution.

PROB. 74*a*.

Let x, y, z denote the base, perpendicular and hypothenuse respectively,

$$\text{then, } x+y+z=24 \ldots\ldots\ldots\ldots (1)$$

$$x=z \cos. 30^\circ=\tfrac{1}{2}\sqrt{3}\cdot z \ldots\ldots (2)$$

$$y=z \sin. 30^\circ=\tfrac{1}{2}z \ldots\ldots\ldots (3)$$

$$\therefore \left(\frac{\sqrt{3}}{2}+\tfrac{1}{2}+1\right)z=24. \quad \therefore z=10{\cdot}144.$$

whence $x=8{\cdot}784 \quad y=5{\cdot}072$.

PROB. 75*a*. See solution of Prob. 79.

PROB. 76*a*.

(By formula). $bc \sin.^2 \frac{A}{2}=\tfrac{1}{2}(a+b-c).\tfrac{1}{2}(a-b+c)$.

Or substituting

$$\left.\begin{array}{ll} (1)\ldots a+b & =600 \\ c & =400 \\ A & =60^\circ \\ \sin.^2 \frac{A}{2} & =\frac{1}{4} \end{array}\right\} \begin{array}{l} 400b.\tfrac{1}{4}=\tfrac{1}{2}(200).\tfrac{1}{2}(a-b+400) \\ \therefore 400b=200.\overline{a-b}+80000 \end{array}$$

$$\text{or } 2b=a-b+400$$

$$\therefore 3b-400=a=600-b \ldots \text{by (1)}$$

$$\therefore 4b=1000$$

$$\text{and } b=250$$

$$\therefore a=350.$$

Prob. 77*a*.

Let z represent the hypothenuse and x the perpendicular.

then $z+x+8=24$........ (1)

and $z^2=x^2+64$.. ....(2)

whence $z=10$ $x=6$.

Prob. 78*a*, (fig. 65).

Let height of steeple BC $=x$

distance AB of 1st station from steeple $=80$

distance DB of 2nd station .... $=250$

Let angle CAB $=2a$, then angle D $=a$

since angle CAB $=$ angle D $+$ angle ACD,

$\therefore 2a=a+$ angle ACD

$\therefore$ angle ACD$=a$; or triangle ADC is isosceles, and AC $=$ AD $=170$,

hence $x=\sqrt{AC^2-AB^2}=\sqrt{170^2-80^2}=150$.

Prob. 79*a*.

Let x and y denote the sides

then $x+y=b$......... (1).

and (by formula). $xy\cos.^2\frac{A}{2}=\frac{1}{2}(x+y+a).\ \frac{1}{2}(x+y-a)$

$$=\tfrac{1}{4}.\ \overline{b+a}.\ \overline{b-a}$$

$\therefore xy=\frac{1}{4}(b+a).(b-a).\ \sec.^2\frac{A}{2}$ .... (2).

From equations (1) and (2), x and y are easily deduced.

PROB. 80*a*.

Let x and y denote the sides. $\therefore\ x-y=d \ldots\ldots (1)$

and by formula $xy \sin.^2 \frac{A}{2} = \frac{1}{2}(a+\overline{x-y}) . \frac{1}{2}(a-\overline{x-y})$

$$= \frac{1}{4}(a+d).(a-d).$$

$$\therefore\ xy = \frac{1}{4}(a+d).(a-d) \operatorname{cosec}^2 \frac{A}{2}.$$

PROB. 81*a*.

Let A denote the angle opposite base a,
x and y the angles at the base,
and $x_{,}$ and $y_{,}$ the sides opposite to x and y,
then $x_{,}+y_{,}=b \ldots\ldots\ldots\ldots\ldots (1)$,
and $x-y=d$.

Now $\frac{\sin. x}{\sin. A} = \frac{x_{,}}{a}$

$\frac{\sin. y}{\sin. A} = \frac{y_{,}}{a}$

adding, $\frac{\sin. x + \sin. y}{\sin. A} = \frac{x_{,}+y_{,}}{a} = \frac{b}{a}$

or $\frac{\sin. x + \sin. y}{\sin. (x+y)} = \frac{b}{a}$ since $A = 180 - \overline{x+y}$

or $\frac{2 \sin \frac{1}{2}(x+y) \cos. \frac{1}{2}(x-y)}{2 \sin \frac{1}{2}(x+y) \cos. \frac{1}{2}(x+y)} = \frac{b}{a}$

or $\frac{\cos. \frac{1}{2} d}{\cos. \frac{1}{2}(x+y)} = \frac{b}{a}$

$$\therefore\ \cos. \frac{1}{2}(x+y) = \frac{a}{b} \cos. \frac{1}{2} d.$$

PROB. 82*a*.

Let a be the base and $x_{\prime}$ $y_{\prime}$ the two sides of triangle.
x, y the angles opposite to $x_{\prime}$ $y_{\prime}$ respectively.
and A the angle opposite base a.

$$\text{then } x_{\prime}-y_{\prime}=d \text{ and } x-y=\text{D}.$$

$$\frac{\sin. x}{\sin. \text{A}}=\frac{x_{\prime}}{a}, \quad \frac{\sin. y}{\sin. \text{A}}=\frac{y_{\prime}}{a}$$

$$\therefore \text{ subtracting } \frac{\sin. x-\sin. y}{\sin. \text{A}}=\frac{x_{\prime}-y_{\prime}}{a}=\frac{d}{a}$$

$$\text{or } \frac{2\cos. \frac{1}{2}(x+y)\sin. \frac{1}{2}(x-y)}{\sin. (x+y)}=\frac{d}{a}$$

$$\text{or } \frac{2\cos. \frac{1}{2}(x+y)\sin. \frac{1}{2}(x-y)}{2\sin. \frac{1}{2}(x+y)\cos. \frac{1}{2}(x+y)}=\frac{d}{a}$$

$$\text{whence } \sin. \tfrac{1}{2}(x+y)=\frac{a}{d}\sin. \tfrac{1}{2}(x-y)=\frac{a}{d}\sin. \tfrac{1}{2}\text{D}$$

and since $x-y$ is also known, the sides and angles of triangle may be found.

PROB 83*a*.

Let a, b, c be the three sides of the triangle of which the base is b.

$$\text{then } a-c=d \text{ ....} (1)$$

$$(\text{By formula}) \tan. \frac{\text{A}}{2}=\sqrt{\frac{(S-b)\cdot(S-c)}{\text{S}.(S-a)}}$$

$$\text{and } \tan. \frac{\text{C}}{2}=\sqrt{\frac{\overline{S-a}.\overline{S-b}}{S\cdot\overline{S-c}}}$$

$$\therefore \frac{\tan.\frac{C}{2}}{\tan.\frac{A}{2}} = \frac{S-a}{S-c} = \frac{\frac{a+b+c}{2}-a}{\frac{a+b+c}{2}-c} = \frac{b+c-a}{b+a-c}$$

$$= \frac{b-(a-c)}{b+a-c} = \frac{b-d}{b+d} \ldots\ \text{from (1)}$$

$$\therefore \tan.\frac{C}{2} = \frac{b-d}{b+d}\tan.\frac{A}{2}.$$

PROB. 84a.

Let $a+c = m \ldots\ldots (1)$

$$\tan.\frac{A}{2} = \sqrt{\frac{\overline{S-b}\cdot\overline{S-c}}{S\cdot\overline{S-a}}}$$

$$\tan.\frac{C}{2} = \sqrt{\frac{\overline{S-a}\cdot\overline{S-b}}{S\cdot\overline{S-c}}}$$

$$\therefore \tan.\frac{A}{2}\tan.\frac{C}{2} = \frac{S-b}{S} = \frac{\frac{a+b+c}{2}-b}{\frac{a+b+c}{2}} = \frac{a+c-b}{a+b+c}$$

$$= \frac{m-b}{m+b} \therefore \tan.\frac{C}{2} = \frac{m-b}{m+b}\cot.\frac{A}{2}.$$

PROB. 85a.

(By formula) $\sin.\frac{1}{2}A = \sqrt{\frac{(S-b).(S-c)}{bc}}$

$$\cos.\tfrac{1}{2}B = \sqrt{\frac{S.(S-b)}{ac}} \text{ and } \cos.\tfrac{1}{2}C = \sqrt{\frac{S.(S-c)}{ab}}$$

$$\frac{\sin.\tfrac{1}{2}A}{\cos.\tfrac{1}{2}B\ \cos.\tfrac{1}{2}C} = \frac{a}{S}$$

whence $a = S \sin.\tfrac{1}{2}A \sec.\tfrac{1}{2}B \sec.\tfrac{1}{2}C.$

PROB. 86*a*.

In fig. (21) produce BCD to a point E, and join AE. Let AB represent the object, and C, D, E the three stations.

Let $DC = a$, $ED = b$, $AB = x$, $AC = z$, $AD = z_{,}$, the angle $ACB = 2\theta$, $ADC = 90 - \theta$, and $AED = \theta$.

$$\text{In tri. AED. } \frac{z_{,}}{b} = \frac{\sin.\theta}{\sin.\text{EAD}} = \frac{\sin.\theta}{\sin.(90-2\theta)} = \frac{\sin.\theta}{\cos.2\theta} \quad \ldots (1)$$

$$\ldots\ \text{ADC}.\ \frac{z_{,}}{a} = \frac{\sin.2\theta}{\sin.\text{DAC}} = \frac{\sin.2\theta}{\sin.(2\theta - 90^\circ + \theta)}$$

$$= \frac{\sin.2\theta}{\sin.(3\theta - 90^\circ)} \quad \ldots (2)$$

$$\ldots\ \text{ADC}.\ \frac{z_{,}}{z} = \frac{\sin.2\theta}{\sin.(90-\theta)} = \frac{\sin.2\theta}{\cos.\theta} \quad \ldots (3)$$

$$\therefore \text{ by } (3) \ldots z = \frac{z' \cos.\theta}{\sin.2\theta} = (\text{by } 1)\frac{b \sin.\theta \cos.\theta}{\sin.2\theta \cos.2\theta} = \frac{b}{2\cos.2\theta}$$

but $z = a + b$ since the triangle AEC is manifestly isosceles.

$$\therefore a + b = \frac{b}{2\cos.2\theta} \text{ or } \cos.2\theta = \frac{b}{2(a+b)}$$

To find x .. $x = z \sin.2\theta = (a+b)\sqrt{1-\cos.^2 2\theta}$

$$= (a+b)\sqrt{\left(1 - \frac{b^2}{4(a+b)^2}\right)} = \tfrac{1}{2}\sqrt{4(a+b)^2 - b^2} =$$

38.73 (in the example)

PROB. 87*a*, (fig. 38).

In triangles, CDB, CDA, $\dfrac{a \text{ sin. } y}{\text{sin. } \alpha} = \text{CD} = \dfrac{b \text{ sin. } x}{\text{sin. } \beta}$

$$\therefore \frac{\text{sin. } x}{\text{sin. } y} = \frac{a \text{ sin. } \beta}{b \text{ sin. } \alpha}$$

$$\text{or } \frac{\text{sin. } x + \text{sin. } y}{\text{sin. } x - \text{sin. } y} = \frac{a \text{ sin. } \beta + b \text{ sin. } \alpha}{a \text{ sin. } \beta - b \text{ sin. } \alpha}$$

$$\therefore \text{(by trig.) } \frac{\text{tan. } \frac{1}{2}(x+y)}{\text{tan. } \frac{1}{2}(x-y)} = \frac{a \text{ sin. } \beta + b \text{ sin. } \alpha}{a \text{ sin. } \beta - b \text{ sin. } \alpha}$$

$$\text{whence tan. } \tfrac{1}{2}(x-y) = \frac{1 - \dfrac{b \text{ sin. } \alpha}{a \text{ sin. } \beta}}{1 + \dfrac{b \text{ sin. } \alpha}{a \text{ sin. } \beta}} \cdot \text{tan. } \tfrac{1}{2}(x+y)$$

$x + y$, the sum of the angles is known $(=360^\circ - \text{C} - \alpha - \beta)$ and by the equation $x - y$ is given, hence the angles x and y may be found.

Applying this expression to an example, suppose that $\alpha = 54^\circ\ 13'\ 45''$, $\beta = 33^\circ\ 52'$, $\text{C} = 77^\circ\ 42'\ 24''$; and that log. $a = 4{\cdot}170262$, log. $b = 4{\cdot}021189$.

Find in the first place the value of $\dfrac{b \text{ sin. } \alpha}{a \text{ sin. } \beta} = 1{\cdot}0329$.

$$\therefore \tfrac{1}{2}(x-y) = \frac{-0.0329}{2.0329} \text{ tan. } \tfrac{1}{2}(x+y).$$

$$\text{or } \overset{+}{\text{tan.}}\ \tfrac{1}{2}(x-y) = \frac{-329}{20329} \overset{-}{\text{tan.}}\ 97^\circ\ 5'\ 55''$$

(placing the proper sign over tan. $97^\circ\ 5'\ 55''$, and thus

determining the sign of tan. $\frac{1}{2}(x-y)$ see Art. 34, p. 33).

$$
\begin{array}{lrrr}
\text{whence } \frac{1}{2}(x-y) = & 7^\circ & 24' & 24'' \\
\frac{1}{2}(x+y) = & 97 & 5 & 55 \\
\hline
\therefore x = & 104 & 30 & 19 \\
y = & 89 & 41 & 31
\end{array}
$$

All the angles of the triangles ACD and CDB being now known, it will be easy to calculate the distances AD, CD, BD. If we perform the operation we shall find AD = 12516, CD=18241 and AD = 10742·5.

SOLUTIONS OF ASTRONOMICAL PROBLEMS.

The Problems in the following pages, in which the heavenly body is supposed to be either in the horizon, or six hours from the meridian, or on the prime vertical, (see prob. 96, 97, 99, &c.) are solved by means of the formulæ in page 98.

These formulæ may be proved as follows:

In fig. *b*, let NWSE represent the horizon of the spectator, Z his zenith, NZS the celestial meridian, WZE the prime vertical, P the north pole of the heavens and WQE the celestial equator. (see def. p. 94).

First. Suppose the heavenly body to be in the horizon, as at D:

then (1) $\cos h = -\tan. l \tan. d$

(2) $\sin. d = \cos l \sin. m$

where h = hour angle of heavenly body

d = its declination

m = its amplitude DW or angle DZW

and l = latitude of spectator.

Join PD and ZD : then in the triangle ZPD,

angle ZPD=hour angle of heavenly body=h

PD=polar distance=$90°-d$

PZ=co-latitude of spectator=$90°-l$

angle PZD=co-amplitude or $90°-$DZW$=90°-m$

and ZD=$90°$.

In quadrantal triangle PZD .... (Rule XVI)

$$\cos. \text{ZPD} = -\cot. \text{PZ} \cot. \text{PD}$$

$$\text{or } \cos. h = -\cot. (90-l) \cot (90-d)$$

$$= -\tan. l \tan. d. \ldots\ldots (1).$$

In same triangle, cos. PD=sin. PZ cos. PZD

$$\text{or } \cos. (90°-d) = \sin. (90°-l) \cos. (90°-m)$$

$$\therefore \sin. d. = \cos l \sin. m. \ldots (2).$$

Second. Suppose the heavenly body to be 6 hours from the meridian as at V, (fig. b).

then (3) $\sin. a = \sin. l \sin. d.$

(4) $\cos. l = \cot. d \cot. Z_{,}$

where a=altitude of heavenly body, $Z_{,}$=its azimuth.

l and d the lat. and decl. as before.

Join PV and ZV : then in triangle ZPV

PZ $= 90°-l$, PV $= 90°-d$,

ZV $=$ zenith distance $= 90°-D_{,}V = 90-a$

angle PZV $=$ azimuth $=Z_{,}$ and ZPV $=$ 6^h or $90°$.

In right angled triangle ZPV .... (Rule XV)

$$\cos. \text{VZ} = \cos. \text{PV} \cos. \text{PZ}$$

$$\text{or } \cos. (90 - a) = \cos. (90 - d) \cos. (90 - l)$$

$$\therefore \sin. a = \sin. d \sin. l \ldots\ldots\ldots (3)$$

In same triangle, sin. PZ = tan. PV tan. PZV

$$\text{or } \sin. (90 - l) = \tan. (90 - d) \tan. Z_{,}$$

$$\therefore \cos. l = \cot. d \tan. Z_{,} \ldots\ldots\ldots (4)$$

Third. Suppose the heavenly body on the prime vertical, as at Y.

$$\text{then (5) } \sin. d = \sin. l \sin. a_{,}$$

$$\text{(6) } \cos. h_{,} = \cos. l \tan. d$$

where $a_{,}$ = altitude, $h_{,}$ = hour angle } when on prime vertical.

and l and d the latitude and declination as before. Join PY: then in triangle ZPY, PZ $= 90° - l$, PY $= 90 - d$, ZY = zenith dist. $= 90 - a_{,}$, angle ZPY = hour angle $= h_{,}$, and PZX = 90°.

In right angled triangle PZY (Rule XV.)

$$\cos. PY = \cos. PZ \cos. ZY$$

$$\text{or } \cos. (90 - d) = \cos. (90 - l) \cos. (90 - a_{,})$$

$$\therefore \sin. d = \sin. l \sin. a_{,} \ldots\ldots\ldots\ldots\ldots (5)$$

In same triangle, cos. ZPY = tan. PZ cot. PY

$$\text{or } \cos h_{,} = \tan. (90 - l) \cot. 90 - d)$$

$$\therefore \cos. h_{,} = \cot. l \tan. d \ldots\ldots\ldots\ldots\ldots (6)$$

In the above formulæ the latitude and declination have been supposed to be of the same name, that is, both north or both south. Formulæ (1) and (2) may be adapted to problems in which the latitude and declination are of different names by putting $90 + d$ for $90 - d$ and $90 + m$ for $90 - m$ in the investigation: we shall then have (p. 50.)

$$\cos. h = - \cot. (90 - l) \cot. (90 + d)$$

$$= - \tan. l \cdot (- \tan. d)$$

$$= \tan. l \tan. d. \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\cos. (90 + d) = \sin. (90 - l) \cos. (90 + m)$$

$$\text{or } - \sin. d = \cos. l. (- \sin. m)$$

$$\therefore \sin. d = \cos. l \sin. m. \text{ (as before)} \ldots. (2_{,})$$

Formulæ (3) (4) (5) (6) require no alteration since they are applied only to problems in which the latitude and declination are of the same name.

PROB. 88, (fig. *b*).

Describe the following parts of fig. *b*; the horizon NWSE, celestial meridian NZS, and prime vertical WZE: then Z is the zenith of the spectator, (see definitions p. 94). Take a distance S*m* on the meridian = 70° (ZS being 90°) to represent the sun's meridian altitude. (The altitude S*m* is measured from S, the south point of the horizon, because by the question, the zenith is north of the sun). Take *m*Q = 20° measured from *m* towards S; then Q will represent the point in the meridian through which the celestial equator WQE passes, and arc *m*Q is the sun's declination = 20° N. The distance *m*Q is taken from *m* towards S the south point of horizon in order that the heavenly body may be to the *north* of the equator or have north declination.

Now ZQ = latitude of place . . . . (p. 94)
$= Zm + mQ = 90° - mS +$ sun's decl.
$= 90° - 70° + 20° = 40°$ N. ans.

If a distance QZP = 90° be taken on the meridian, then P is the north pole; and since Z, the zenith of the place is by the figure north of the equator, the place is in north latitude.

PROB. 89, (fig. 66).

Describe the horizon NWSE, the celestial meridian NZS and prime vertical WZE. Take a distance S*m* on

the meridian $= 70°$ to represent the meridian altitude of the sun: (this must be measured from S upwards since the zenith is north of the sun): and because the declination is south, take mQ $= 5°$ measured from m upwards towards the north, and through Q draw WQE to represent the celestial equator.

Now lat. (ZQ)=Zm — mQ=90° — sun's alt.—sun's decl.
$= 90° — 70° - 5° = 15°$ N.

The lat. is N. because Z is north of the equator.

Prob. 90, (fig. 67).

Describe as before the horizon, celestial meridian and prime vertical. Measure Nm on the meridian $= 70°$ from the point N towards Z (since by the question the zenith is *south* of the body). Again from m downwards, or to the south, take a distance mZQ $= 25°$: through Q draw the celestial equator WQE: then arc mQ represents the *north* declination of star.

Now lat. (ZQ) $= m$Q $-$ mZ $=$ star's decl.—star's zen. dist. $= 25° - 20° = 5°$ N.

Prob. 91, (fig. 68).

Describe horizon, celestial meridian, and prime vertical. Take Nm $= 30°$ to represent the sun's meridian altitude, and mQ$= 10°$ towards Z to represent its declination (10° N.): through Q draw the curve WQE the celestial equator.

Then lat. (ZQ) $= Zm_{,} - mQ$
$=$ sun's zenith distance—sun's decl.
$= 60° - 10° = 50°$ S.

The latitude is *south* since Z is, by the figure, south of the equator, or in the southern hemisphere.

From the last four problems is derived the Rule in Navigation for finding the latitude by the meridian altitude of a heavenly body.

PROB. 92, (fig. 69).

Draw the horizon and celestial meridian, take $Nm =$ $20°$ and $Nm_{,} = 70°$: the former representing the star's meridian altitude *below* the pole, or at its inferior transit, and the latter its meridian altitude at its superior transit. Bisect $mm_{,}$ in P and describe a small circle passing through m and m': this will represent the star's parallel of declination, and P the pole of the heavens, about which the heavenly body appears to move: hence $Pm = Pm_{,} =$ its polar distance. Also PN or the altitude of the pole above the horizon is equal to the latitude of the place.*

Now $PN = Nm + Pm =$ alt. below pole $+$ ✱ pol. dist.
also $PN = Nm_{,} - Pm_{,} =$ alt. above pole $-$ ✱ pol. dist.
adding, 2 PN or 2 lat. $=$ alt. above pole $+$ alt. below pole $= 70° + 20° = 90°$ $\therefore$ lat. $= 45°$

* This may be proved as follows: In (fig. *b*) P represents the pole of the heavens WQE the celestial equator, ZQ is the latitude of the place whose zenith is Z (p. 94), and PN is the altitude of the pole P above the horizon.

Now $PN + PZ = 90° = PQ = PZ + ZQ$; taking away PZ from both sides .... $PN = ZQ$
or, altitude of pole $=$ latitude of place.

Prob. 93. (fig. 70).

Describe horizon and celestial meridian: take $Nm = 20° =$ star's meridian altitude at inferior transit, and since it passes through the zenith Z at its superior transit bisect Zm in P: then P will be the pole of the heavens: and (as in last problem).

PN $= Nm + Pm =$ alt. below pole + ✱ pol. dist.

PN $=$ NZ $-$ PZ $=$ alt. above pole $-$ ✱ pol. dist.

$\therefore$ 2 PN or 2 lat. $=$ alt. below pole $+$ alt. above pole.
$= 20° + 90° = 110°$ $\therefore$ Lat. $= 55°$.

Prob. 94.

Let $x =$ altitude at inferior transit, or below pole.

$\therefore$ $x =$ zenith distance at superior transit.

$\therefore$ $90 - x =$ alt at superior transit.

But (by last prob.) 2 lat. $=$ alt. above pole $+$ alt. below pole. $\therefore$ 2 lat.$= x + 90 - x = 90°$ $\therefore$ lat.$= 45°$.

Prob. 95. (fig. 71).

Describe the horizon, celestial meridian, prime vertical and also WQE the celestial equator. Take some point X on the equator to represent the sun's place and draw circle of declination PX and circle of altitude ZO.

Let $x =$ lat. required $= 90° -$ PZ

$a =$ XO, alt. of sun, and $h =$ ZPX the hour angle: then PZ $= 90° - x$, and ZX $= 90° - a$: also PZX is a quadrantal triangle, since PX $= 90°$.

(By Rule XVI) .... $\cos. ZX = \sin. PZ \cos. ZPX$
or $\cos. (90 - a) = \sin. (90 - x). \cos. h$

$\therefore \sin. a = \cos. x \cos. h$, or $\cos. x = \dfrac{\sin. a}{\cos. h} = \sin. a \sec. h$.

Calculation

log. sin. a....	9·590686	
.. sec. h....	0·150515	
.. cos. x....	9·741201	$\therefore x = 56^\circ\ 33'\ 30''$ N.

PROB. 96.

Describe the following parts of fig. (b) : horizon, celestial meridian, prime vertical, and celestial equator. Let D be the place of heavenly body when setting : through D draw circle of decl. PD and circle of altitude ZD. Then PZD is a quadrantal triangle, and we have given PD the polar dist. $= 90^\circ - d$, angle $PZD = 90^\circ - DZW = 90^\circ - m$: to find $PZ = 90^\circ - l$.

By Rule XVI. $\cos. PD = \sin. PZ \cos. PZD$.

or $\cos. (90 - d) = \sin. (90 - l) \cos. (90 - m)$

$\therefore \sin. d = \cos. l. \sin. m$.. (see also form. 2. p. 50.)

or $\cos. l = \sin. d \operatorname{cosec} m$. $\therefore l = 64^\circ\ 29'\ 15''$ N.

PROB. 97.

In triangle PVZ (fig. b) there are given $ZV = 90^\circ - a$, $PV = 90 - d$ and angle $VPZ = 6$ hours $= 90^\circ$: to find co-lat. $PZ = 90 - l$.

(By Rule XV or form. 3. p. 50) $\cos. (90 - a) = \cos. (90 - l) \cos.(90 - d)$ or $\sin. a = \sin. l \sin. d$.

$\therefore \sin. l = \sin. a \operatorname{cosec} d$, whence $l = 69^\circ\ 31'\ 40''$ N.

PROB. 98.

In triangle PVZ (fig. b) are given PZ $= 90° - l$ PV $= 90° - d$, and VPZ $= 90°$: to find ZV $= 90° - a$ and azimuth PZV $= Z_{,}$.

(By p. 50) form. 3. 4. $\left.\begin{array}{l} \sin. a = \sin. l \sin. d \quad \therefore \quad a = 17° 58' 15'' \\ \cos. l = \cot d \cot Z_{,} \therefore Z_{,} = 74° 39' 30''. \end{array}\right.$

These formulæ are also easily obtained by Rule XV as in last prob.

PROB. 99.

In triangle PZY (fig. b) are given PY $= 90 - d$, PZ $= 90 - l$ and PZY $= 90°$: to find ZY $= 90 -$ alt. or $90 - a_{,}$ and hour angle ZPY $= h_{,}$ (see page 51) then by Rule XV, (or by p. 51) $\cos. h_{,} = \cot. l \tan. d$ and $\sin. d = \sin. l \sin. a_{,}$.

whence $a_{,} = 30° 55'$ and $h_{,} = 4^h 37^m 4^s$.

PROB. 100.

In triangle PZD (fig. b) are given PZ $= 90° - l$, PD $= 90 - d$ and ZD $= 90°$: to find PZD $= 90° - m$ and hour angle ZPD $= h$.

By Rule XVI or by p. 50.

$\sin. d = \cos. l \sin. m \quad \therefore \quad m = 30° 4' 30''$

$\cos. h = - \tan. l \tan. d \quad \therefore \quad h = 7^h 36^m 41^s$

(In the last formula $\cos. h$ is negative or h is greater than 90° or greater than 6^h (see Ex. 128 in Part I.)

Whence sun sets at $7^h 36^m 41^s$ and rises at $12^h - 7^h 36^m 41^s$ or at $4^h 23^m 19^s$ and length of day $= 2. 7^h 36^m 41^s = 15^h 13^m 22^s$.

PROB. 101. (fig. 72.)

Suppose the heavenly body to rise at S and to pass the prime vertical at X then PS = PX the star's polar distance; and EX = star's altitude when due east = 20°. Let $a_{,}$ = alt. at X; m = amplitude ES = 11° 15′ x = latitude of spectator, and y = decl. of star.

Then PS = PX = $90 - y$; PZ = $90 - x$ and ZX = $90 - a_{,}$

in quad. tri. PZS.. (Rule XVI or form. 2 p. 50)

$$\sin. y = \cos. x \sin. m \ldots\ldots\ldots\ldots (1)$$

In right ang. tri. PZX .... (Rule XV or form. 5. p. 51)

$$\sin. y = \sin. x \sin a_{,} \ldots\ldots (2)$$

∴ Equating (1) and (2), $\sin. x \sin. a_{,} = \cos. x \sin. m.$

or $\dfrac{\sin. x}{\cos. x.} = \dfrac{\sin. m}{\sin. a_{,}}$ ∴ $\tan. x = \sin. m \operatorname{cosec.} a_{,}$

$= \sin. 11° 15' \operatorname{cosec} 20°$ ∴ $x = 29° 42'$ N.

PROB. 102. (fig. *b*.)

Let x = lat, $a_{,}$ = alt. when due west, and d = decl.

In tri. PZY let Y be the place of star: then PZ = $90 - x$, PY = $90 - d$ and ZY = $90 - a_{,}$.

(By Rule XV. or form. 5 p. 51.) $\sin. d = \sin. x \sin. a_{,}$

∴ $\sin. x = \sin. d \operatorname{cosec.} a_{,} = \sin. 20° \operatorname{cosec.} 30°$

∴ $x = 43° 9' 30''$.

PROB. 103. (fig. 73).

Describe horizon, celestial meridian, prime vertical, and equator. Place X the heavenly body above the equator, since the declination is supposed to be north:

and draw circle of declination PX and circle of altitude ZX. Then in spherical triangle PZX the three sides are given (namely co-latitude PZ $= 90° - 50° \ 48'$, zenith distance ZX $= 90° - 46° \ 20'$ and polar distance PX $= 90° - 23° \ 27' \ 45''$), to find the hour angle ZPX and azimuth PZX.

Calculation. (Rule VIII or IX).

Given	To find az. PZX	To find time ZPX.
ZX=43° 40′ 0″	39° 12′ 0″	66° 32′ 15″
PZ=39 12 0	43 40 0	39 12 0
PX=66 32 15		
	4 28 0	27 20 15
	66 32 15	43 40 0
	71 0 15	71 0 15
	62 4 15	16 19 45
	0·199263	0·037479
	0·160860	0·199263
	4·763976	4·763976
	4·712286	4·152340
	9·836385	9·153058
	111° 51′	$2^h \ 57^m \ 16^s$

hence the azimuth or bearing of the body is 111° 51′ reckoning from the north towards the west, or N. 111° 51′ W.: and the apparent time or hour angle from noon is $2^h \ 57^m \ 16^s$.

(From this problem the Rules in Navigation for finding the azimuth of a heavenly body, and the time at the ship, or place of observation are derived).

PROB. 104, (fig. 74).

(1st Sol.) Describe fig. 72, and draw XM a perpendicular on meridian ZM : then (by Rule XV).

1. In tri. XZM.... sin. XM = sin. ZX. sin. MZX
=cos. a sin. az.

2. In tri. XPM .... cos. PX=cos. PM, cos. XM
or cos. PM=sin. d sec. XM.

3. In tri. XZM .... cos az.=tan. alt. tan. ZM,
or tan. ZM=cos. az. cot. a.

∴ whence XM = 38° 34′ 30″; PM = 61° 8′ 45″. and ZM=38° 12′ 45″.

Subtracting ZM from PM we have, PZ=30° 56′, and ∴ lat. = 59° 4′ N.

(2nd Sol.) (fig. 73) without dropping a perpendicular.

3. In tri. PZM.... $\frac{\text{sin. P}}{\text{sin. PZX}} = \frac{\text{sin. ZX}}{\text{sin. PX}}$ .. (Rule XII)

∴ sin. P=sin. PZX, sin. ZX cosec. PX.
=sin. 57° 45′ cos. 42° 30′ sec. 22° 10′ ∴ P=42° 19′ 30″.

Then, having given two sides and two opposite angles, namely, the polar dist. PX, zen. dist. ZX, azimuth PZX, and hour angle P, we have by Napier's Analogies,

$$\text{tan. } \tfrac{1}{2} \text{ PZ} = \frac{\text{tan. } \frac{1}{2} \text{ (PX}-ZX) \text{ sin. } \frac{1}{2} (\text{P}ZX+\text{P})}{\text{sin } \frac{1}{2} (\text{P}ZX-\text{P})}$$

$= \text{tan } \frac{1}{2}(\text{PX}-ZX) \text{ sin. } \frac{1}{2} (\text{P}ZX+\text{P}) \text{ cosec } \frac{1}{2} (\text{P}ZX-\text{P})$
whence $\frac{1}{2}$ PZ=15° 27′ 45″ and lat. = 59° 4′ 30″.

(3rd Sol.) without first finding P, by using a subsidiary angle.

$$\text{Since cos. PZX} = \frac{\text{cos. PX} - \text{cos. PZ cos. ZX}}{\text{sin. PZ sin. ZX}}$$

$$= \frac{\text{sin. } d - \text{sin. } l \text{ sin. } a}{\text{cos. } l \text{ cos. } a}.$$

$$\therefore \text{sin. } d = \text{sin. } a \text{ sin. } l + \text{cos. } l \text{ cos. } a \text{ cos. PZX}$$

$$= \text{sin. } a \text{ (sin. } l + \text{cos. } l \text{ cot. } a \text{ cos. PZX)}$$

Assume $\text{cot. } \theta = \text{cot. } a \text{ cos. PZX}$

$$\therefore \text{sin. } d = \text{sin. } a \text{ (sin. } l + \text{cos. } l \text{ cot. } \theta)$$

$$= \text{sin. } a \text{ cosec } \theta \text{ (sin. } l \text{ sin. } \theta + \text{cos. } l \text{ cos. } \theta)$$

$$= \text{sin. } a \text{ cosec } \theta \text{ cos. } (l \backsim \theta)$$

$\therefore$ cos. $(l \backsim \theta) =$ sin. d cosec a sin. θ : from which $l \backsim \theta$, and consequently l may be obtained.

Calculation

$\text{cot. } \overset{-}{\theta} = \text{cot. } \overset{+}{a} \text{ cos. } \overset{-}{\text{PZX}}$ $\text{cos. } (l \backsim \theta) = \text{sin. } d \text{ cosec } a \text{ sin. } \theta$

cot. a .. 0·037948	cosec a .. 0·170317
cos. Z .. 9·727228	sin. θ .. 9·936597
	sin. d .. 9·576689
cot. θ .. 9·765176	
59° 47′ 10″	cos. $(\theta \backsim l)$ .. 9·683603
180	$\therefore$ $\theta - l$ .. 61° 8′ 30″
$\therefore$ θ .. 120 12 50	

whence $l = \theta - 61° \; 8' \; 30'' = 59° \; 4' \; 20''$.

PROB. 105. (fig. 74).

(1st Sol.) In PMX.. cos. h = tan. PM tan. d

$\therefore$ tan. PM = cos. h cot. d .... (1)

In tri. PMX.. $\sin. MX = \sin. h \cos. d$ .... (2)
In tri. ZMX.. $\cos. ZX = \cos. ZM \cos. MX$

$\therefore \cos. ZM = \sin. a \sec. MX$ ...... (3)

From (1) PM $= 63° 31' 15''$, from (2) MX $= 30° 53'$, from (3) ZM $= 45° 2' 15''$: subtracting (3) from (1) ZP $= 18° 29'$ whence lat. $= 71° 31'$.

(2nd Sol.) fig. 73 (without dropping a perpendicular)

$$\frac{\sin. PZX}{\sin. P} = \frac{\sin. PX}{\sin. ZX} \therefore \sin. PZX = \cos. d \sin. h \sec \text{alt}.$$

$\therefore$ PZX $= 180° - 40° 12' 30'' = 139° 47' 30''$ (the angle found in table is in this case taken from 180°, since the angle PZX by construction is greater than 90°:) then, as in last prob. the value of $\frac{1}{2}$ PZ may be found ($= 9° 14' 20''$) whence lat. $= 71° 31' 20''$.

(3rd Sol.) as in last problem, (using a subsidiary angle)

$$\cos. P = \frac{\cos. ZX - \cos PX \cos. PZ}{\sin. PX \sin PZ} = \frac{\sin a - \sin. d \sin. l}{\cos. d \cos. l}$$

$$\therefore \sin. a = \sin. d \sin. l + \cos. d \cos. l \cos. P$$
$$= \sin. d (\sin. l + \cos. l \cot. d \cos. P)$$

Assume $\cot. \theta = \cot. d \cos. P$,

$$\therefore \sin. a = \sin. d (\sin. l + \cos. l \cot. \theta)$$
$$= \sin. d \operatorname{cosec} \theta \cos. (l \sim \theta)$$

whence $\cos. (l \sim \theta) = \sin. a \operatorname{cosec} d \sin \theta$, which determines $l \sim \theta$ and consequently l.

(4th Sol.) In tri. ZPX $\cos. h = \dfrac{\cos. ZX - \cos. PX \cos PX}{\sin. PX \sin. PX}$

$$\frac{\cos. z - \sin. d \sin. l}{\cos. d \cos. l}$$

$\therefore$ cos. d oos. l cos. h = cos. z — sin. d sin. l,

Subtracting both sides from cos. d cos. l, we have

cos. d cos. l (1 — cos. h) =

— cos. z + cos. d cos. l + sin. d sin. l

or cos. d cos. l ver. h = — cos. z + cos. $(d \backsim l)$

$\therefore$ — cos. $(d \backsim l)$ = — cos. z—cos. d cos. l ver. h.

adding 1 to both sides.

1 — cos. $(d \backsim l)$=1—cos. z — cos. d cos. l ver h.

or ver. $(d \backsim l)$ = ver. z — cos. d cos. l ver. h.

This formula determines the value of $d \backsim l$, where d is the declination at the time of the observation, and l the latitude required. In this expression, however, the latitude itself is involved, since without knowing l we cannot find $d \backsim l$: we must therefore use in the first place the estimated latitude of the ship or place of observation, which will give us, in most cases, a near approximation to the true latitude. But if the latitude thus found differ considerably from the estimated latitude the work must be repeated, using the latitude found instead of the estimated latitude.

To reduce ver. $(d \backsim l)$ = ver. z — cos. d cos. l ver. h to logarithms, let us assume

ver. θ = cos. d cos. l ver h = 2 cos. d cos. l $\frac{1}{2}$ ver. h.

or ver. θ = 2 cos. d cos. l haversine h. . p. IV. Part I.

In log. . . . log. ver. θ—6 = log. 2 + log. cos. d + log. cos. l + log. hav. h—30 (see Art. 32. p. 31. Part. I.)

$\therefore$ log. ver. θ = 6·301030 + log. cos. d + log. cos. l + log. hav. h—30 : which expression determines θ;

and ver. $(d \backsim l)$ = ver. z — ver. θ.

whence $d-l$ may be found: and since d the declination at the time of the observation is supposed to be known, the latitude l is determined.

From this solution is derived the Rule in Navigation for finding [the latitude by an altitude taken near the meridian. The problem is usually limited to this position of the body, because when the azimuth is small, an error of 3 or 4 minutes in the time, or hour angle, will not produce any considerable error in the resulting latitude. This may be proved as follows.

(PROB. A). To find the corresponding error in the latitude for a given error in the hour angle of a heavenly body near the meridian (fig. 75)

Let h = SPZ the true hour angle.

z = SZ the true zenith distance.

$h_{,}$ = $S_{,}$PZ the erroneous hour angle.

and $z_{,}$ = $S_{,}$P the zenith distance computed from it.

p = PS = $PS_{,}$ = the pol. dist. and c = colat PZ.

In triangle ZPS ...... $\cos. h = \dfrac{\cos. z - \cos. c \cos. p}{\sin. c \sin. p}$

.. .. $ZPS_{,}$...... $\cos. h_{,} = \dfrac{\cos. z_{,} - \cos. c \cos. p}{\sin. c \sin. p}$

Subtracting ...... $\cos. h - \cos. h_{,} = \dfrac{\cos. z - \cos. z_{,}}{\sin. c \sin. p}$

or $\sin. \frac{1}{2}(h+h_{,}) \sin. \frac{1}{2}(h-h_{,}) = \dfrac{\sin. \frac{1}{2}(z+z_{,}) \sin. \frac{1}{2}(z-z_{,})}{\sin. c \sin. p}$

But since $h=h_{,}$ nearly, and $z=z_{,}$ nearly, we may put $h = \frac{1}{2}(h+h_{,})$, $z = \frac{1}{2}(z+z_{,})$, $\frac{1}{2}(h-h_{,})$ for $\sin. \frac{1}{2}(h-h_{,})$ and $\frac{1}{2}(z-z_{,})$ for $\sin. \frac{1}{2}(z-z_{,})$.

Making these substitutions and cancelling,

$$\text{we have.... sin } h(h-h_{\prime}) = \frac{\sin. z(z-z_{\prime})}{\sin. c \sin. p}$$

Let A = azimuth PZS : then in triangle PZS,

$$\frac{\sin. h}{\sin. A} = \frac{\sin. z}{\sin. p} \text{ or } \sin. h = \frac{\sin. A \sin. z}{\sin. p}$$

Substituting this value of sin. h in the above expression, and cancelling, we have, $z-z_{\prime} = \sin. c \sin. A(h-h_{\prime})$. $z-z_{\prime}$ is the corresponding error in the zenith distance for the given error $h-h_{\prime}$ in the hour angle: but since the heavenly body is supposed to be near the meridian the zenith distance observed is nearly equal to the *meridian* zenith distance ($= l \backsim d$, see figs. 66, 67, &c.): or zen. dist. observed $= l \backsim d$ nearly: therefore an error $z-z_{\prime}$ in the observed zenith distance will produce the same error (nearly) in the latitude, or $z-z_{\prime}$ = error in lat. (nearly): hence

Error in lat. = cos. l sin. A. error in hour angle.

From this formula it appears that the error in the latitude produced by a given error in the hour angle, will increase as azimuth of the heavenly body increases; hence the nearer the body is to the meridian the better: this is seen in the following examples.

Examples. Let the latitude of place = 50° 48′ N. Error in hour angle = 3 min. = 45′ (in arc.). Required the corresponding error in the latitude; (1st) when azimuth = 1 point; (2) when azimuth = 1°.

(1) az. = 1 pt. = $11^{\circ}\ 15'$		(2) az. = 1°	
log. cos. l	9·800737	log. cos. l ..	9·800737
.. sin. $11^{\circ}\ 15'$..	9·290236	.. sin. 1°..	8·241855
.. $45'$........	1·653213	.. $45'$....	1·653213
	0·744186		$\bar{1}$·695805
Err. in lat...	$=5'{\cdot}5$	∴ Err. in lat...	$=0'{\cdot}5$

By means also of the expression found above, we can easily shew that for any given error in the altitude, the error in the hour angle determined by Prob. 103, will be the *least* when the *azimuth* is the *greatest*; that is when the bearing of the body is due east or west: this may be enunciated as follows:

Prop. B. To find the corresponding error in the hour angle for a given error in the observed altitude

(l. 6. p. 65) $(h-h_{,})$ sin. c sin. $A = z - z_{,}$

$$\text{or error in hour angle} = \frac{\text{err. in alt.}}{\sin. c \sin. A}$$

from which it appears that the error in hour angle will be the least when sin. A is the greatest, that is, when the azimuth $=90^{\circ}$

Prob. 106. (fig. 76).

Let X be the place of the sun, ARQ the celestial equator, AX the ecliptic. Through X suppose a circle of declination PXR to be drawn, and produced to cut the equator in R. Then in right angled triangle AXR,

AR, the sun's right ascension, and angle MAR, the obliquity of ecliptic (see definitions, p. 95, 96) are given, to find AX the sun's longitude, and XR its declination.

Ex. 1. (RULE XV).

To find long... cos. A=tan. AR *cot. AX* .. (1)
To find decl... sin. AR=*tan. XR* cot. A.... (2)
from (1) cot. AX=cos. A cot. AR ∴ long.=64° 33′ 15″
(2) tan. XR=sin. RA tan. A ∴ decl.= 21 4 15 N.

Ex. 2. (fig. 77).

In this example, the sun has passed the autumnal equinox, since its right ascension is greater than 12 hours and its declination is therefore south. To find the sun's place in the ecliptic, let $ALA_{,}$ represent the whole of the celestial equator, A and $A_{,}$ the first point of Aries, $ACLC_{,}A_{,}$ the ecliptic, L the first point of Libra : subtract 12 hours from the sun's right ascension and let LR represent the remainder : through R draw a circle of declination cutting the ecliptic in X, then X is the sun's place. By above formulæ

sin RL=cot. ob. *tan. decl.* ∴ decl.=22° 58′ S.
cos. ob.=tan. LR *cot. LX* ∴ LX =78° 30′ 15″
∴ long. ACLX=180°+78° 30′ 15″=258° 30′ 15″.

PROB. 107. (fig. 78.)

Ex. 1. Let X be the place of the heavenly body, AR the celestial equator, AM the ecliptic. Through X

draw the circle of declination PXR and circle of latitude P′XM, and through A and X a great circle AX.

In the figure we have given AR and XR the right ascension and declination of body, and obliquity of ecliptic MAR, to find AM and XM the longitude and latitude.

(1) In rt. ang. tri. AXR.. *cos. AX*=cos. RAcos.XR

(2) .... sin. AR=tan. XR *cot.XAR.*

From equation (1) the arc AX is found ($=48^\circ\ 45'\ 45''$) and from (2) the angle XAR ($29^\circ\ 7'\ 0''$), whence the angle XAM (=XAR∼obliq.)$=5^\circ\ 39'\ 15''$ is known.

(3) In rt. ang. tri. XAM.. cos. XAM=cot.AX.*tan.long.*

(4) .... *sin. lat*=sin.AXsin.XAM.

Ex. 2. (fig. 77).

Let $X_{,}$ be the place of body, L the first point of Libra, $X_{,}R_{,}$ the declination (North) and $X_{,}M$ the latitude of $X_{,}$: join $LX_{,}$. In the fig. we have given $LR_{,}$=star's $RA-12^h=1^h\ 14^m$, and $X_{,}R_{,}$=✻ decl.$=25^\circ\ 51'$ N. and obliquity RLM$=23^\circ\ 27'\ 45''$.

(1) In triangle $LX_{,}R_{,}$.. *cos.* $LX_{,}$=cos. $LR_{,}$ cos $X_{,}R_{,}$.

(2) sin. $LR_{,}$ = *cot.* $X_{,}LR_{,}$ tan. $XR_{,}$, whence $LX_{,} = 66^\circ\ 19'\ 30''$, and $X_{,}LR_{,} = 28^\circ\ 25'\ 45''$.

In tri. $X_{,}LM$, given $LX_{,}$ and $X_{,}LM$ ($=X_{,}LR_{,}$+obliq.) $= 51^\circ\ 53'\ 30''$, to find LM and latitude $X_{,}M$.

(3) sin. $X_{,}M$ = sin. $LX_{,}$ sin. $X_{,}LM$.

(4) cos. $X_{,}LM$ = cot. $LX_{,}$ *tan. LM.*

From (3) $X_{,}M = 46^\circ\ 6'\ 15''$ N. = latitude.

.. (4) LM $= 54^\circ\ 36'\ 30''$, wherefore long. ACLM $= 180^\circ + 54^\circ\ 36'\ 30'' = 234^\circ\ 36'\ 30''$.

PROB. 108. (fig. 79).

Let A and B represent the two places in the same latitude, UV the arc of the equator intercepted between the meridians PAU, PBV passing through the two places: then AU = BV = lat. 45°. ∴ PA = PB = 45°, and angle APB = arc UV = difference of longitude = 10° 36′.

In spherical triangle APB, we have given, PA, PB each = 45° and included angle = 10° 36′, to find AB = 7° 29′ 16″ = (in minutes) 449′·25 the distance in nautical miles.

PROB. 109. (fig. 80).

Let UV represent the terrestial equator, AV the latitude of Portsmouth = 50° 48′ N.: BU the latitude of Buenos Ayres = 34° 37′ S.; PV, PB meridians passing through A and B: draw AB a great circle passing through A and B: then in spherical triangle PBA, PB = 90°+34° 37′ = 124° 37′; PA = 90° − 50° 48′ = 39° 12′, and included angle P = UV = difference of longitude between the places = 57° 18′ whence (Rule X) the third side AB = 99° 9′ 48″ = 5949′.8 nautical miles. To find corresponding distance in English miles we have 60 : 69·05 : : 5949·8 : x = 6847·2 the distance in English miles.

PROB. 110, (fig. 81).

Let AQ represent the celestial equator, A the first

point of Aries, M the place of moon, R the place of star, PM, PR circles of declination passing through M and R, and MR the arc of a great circle joining the two places: then MR is the distance required.

In triangle MPR we have given PM $= 90 +$ moon's decl. $= 95° 19'$, PR $= 90 -$ star's decl. $= 76° 49' 45''$ and included angle P $=$ difference of right ascensions $= 147° 46'$; to find the third side $=$ (by Rule X) $147° 16' 2''$.

In the Rule in Navigation for finding the longitude by *lunar observations*, the true distance MR of the moon from the sun or some other heavenly body is required: in this case we have given the following quantities: the *apparent* and *true* altitudes of each of the heavenly bodies: and their *apparent* distance. Thus, fig. 82, let $S_{,}$ be the apparent place of the sun; then, in consequence of refraction and parallax (the former exceeding the latter) its *true* place (p. 92. P. I.) will be below $S_{,}$ as S. Let $M_{,}$ be the apparent place of the moon, then its *true* place will be above $M_{,}$ since the moon is depressed by parallax considerably more than it is raised by refraction. Let its true place be denoted by M. Draw through $M_{,}$ and $S_{,}$ the great circle $M_{,}S_{,}$: this will be the apparent distance found by observation; draw also the arc MS which will be the true distance to be computed.

Let *Z* be the zenith of the spectator, then, if we suppose the effect of parallax to take place in a vertical circle, $ZM_{,}$ and ZS are circles of altitude.

Let ZM the true zenith dist. of moon $=z$

ZS .... sun or star $=z_{\prime}$

$\text{ZM}_{\prime}$ .. apparent zen. dist. of moon $=90-a$

$\text{ZS}_{\prime}$ .... sun or star $=90-a_{\prime}$

(where a and $a_{\prime}$ are the apparent altitudes of moon and sun or star.)

$\text{M}_{\prime}\text{S}_{\prime}$ apparent dist. of bodies........ $=d$

and MS their true distance .... $=x$

(1st Sol.) The true distance x may be found by the common rules of Spherical Trigonometry (VIII and X).

(1) In triangle $\text{M}_{\prime}\text{ZS}_{\prime}$ the three sides are given, viz. the two apparent zenith distances and the apparent distance: to find the angle $\text{M}_{\prime}\text{ZS}$

(2) In the triangle MZS are given the two sides MZ and SZ the true zenith distances and the included angle Z: to find the third side MS or x.

(2nd Sol.) But the true distance x is more correctly computed by using a subsidiary angle.

In triangle ZMS .... $\cos. Z = \dfrac{\cos. x - \cos. z \cos. z_{\prime}}{\sin. z \sin. z_{\prime}}$

In triangle $\text{ZM}_{\prime}\text{S}_{\prime}$.... $\cos. Z = \dfrac{\cos. d - \sin. a \sin a_{\prime}}{\cos. a \cos. a_{\prime}}$

$$\therefore \quad \frac{\cos. x - \cos. z \cos. z_{\prime}}{\sin. z \sin. z_{\prime}} = \frac{\cos. d - \sin. a \sin. a_{\prime}}{\cos. a \cos. a_{\prime}}$$

adding 1 to both sides, and multiplying up:

$$\frac{\cos. x - (\cos. z \cos. z_{\prime} - \sin. z \sin. z_{\prime})}{\sin. z \sin. z_{\prime}} = \frac{\cos. d - \sin. a \sin. a_{\prime} + \cos. a \cos. a_{\prime}}{\cos. a \cos. a_{\prime}}$$

$$\text{or } \frac{\cos.\, x - \cos.\,(z+z_{,})}{\sin.\, z \sin.\, z_{,}} = \frac{\cos.\, d + \cos.\,(a+a_{,})}{\cos.\, a \cos.\, a_{,}}$$

$$\therefore \cos.\, x - \cos.(z+z_{,}) = (\cos.d + \cos.\overline{a+a_{,}}) \cdot \frac{\sin.\, z \sin.z_{,}}{\cos.\, a \cos\, a_{,}}$$

$$= (\cos.\, d + \cos.\, \overline{a+a_{,}}) \cdot 2 \cos.\, A$$

$$\text{(If we assume } \frac{\sin.\, z \sin.\, z_{,}}{\cos.\, a \cos.\, a_{,}} = 2 \cos.\, A)$$

$$\therefore \cos.x - \cos.(z+z_{,}) = 2 \cos.\, d \cos.A + 2 \cos.(a+a_{,}) \cos.A$$
$$= \cos.\,(d+A) + \cos.(d-A) + \cos.\,(a+a_{,}+A) + \cos.(a+a_{,}-A):$$

transposing cos. $(z+z_{,})$ and subtracting each term from 1 we have.

$$1 - \cos.\, x = 1 - \cos.\,(z+z_{,}) + 1 - \cos.\,(d+A) + 1 - \cos.\,(d-A)$$
$$+ 1 - \cos.\,(a+a_{,}+A) + 1 - \cos.\,(a+a_{,}-A) - 4.$$

or in tabular versines, (see p. 31. P. I.)

$$\therefore \text{tab. ver. } x = \text{tab. ver. } (z+z_{,}) + \text{tab. ver. } (d+A) + \text{tab. ver.} (d-A)$$
$$+ \text{tab. ver. } (a+a_{,}+A) + \text{tab. ver. } (a+a_{,}-A) - 4000000$$

The subsidiary angle A is found in the Nautical tables of Inman and Riddle.

Example.

Required the true distance of the moon from the sun, from the following observation.

App. alt. sun and moon	True zen. dist.
a .... 34° 21′ 32″	z .... 55° 39′ 46″
$a_{,}$ 57 11 25	$z_{,}$ 32 19 50

App. dist..... d.... 35° 47′ 24″		A ... 60° 25′ 16″.	
tab. ver. $(z+z_{\prime})$......	964810	..pts. for ″..	174
ver. $(d+A)$	1107999	....	195
ver. $(d\backsim A)$	90885	....	104
ver. $(a+a_{\prime}+A)$..	1882674	....	30
ver. $(a+a_{\prime}-A)$..	143883	....	102
	4190251		
	605	....	
	4190856		
	4000000		
∴ tab. ver. x....	190856	∴ $x=35°\ 59'\ 16''$.	

In practice it is not necessary to take from the table of versines more than the last five figures, rejecting also all but these five in the sum, since the true distance will be always either in the same column with the apparent distance d, or in the adjacent one.

PROB. 111. (fig. 83).

Project this figure on the plane of the celestial meridian of the place. Let P be the pole, EQ the celestial equator, Z the zenith, S the place of the sun below the horizon Hh, draw circle of declination PS and circle of altitude ZS. (1) In triangle ZPS there are given SP $=90°+$ sun's decl. $=100°\ 15'$, PZ (p. 95. Part I.) the colat. $=39°\ 12'$ and hour angle P $=7^{h}$, to find the third side SZ $=107°\ 24'$: whence $107°\ 24'-90°=17°\ 24'$ the sun's depression.

(2) In the same triangle the three sides are now known, hence the angle PZS the azimuth may be found = N 84° 53′ W.

PROB 112. (fig. 84).

Let AQ represent the celestial equator, AR the ecliptic, S the place of sun in ecliptic, and M the place of moon. Let P′ be the pole of the ecliptic, and P′M, P′S circles of latitude passing through the two bodies: draw SM an arc of a great circle. Then in the quadrantal triangle MP′S there are given MP′ = 90° − moon's latitude = 85° 5′ 30″, PS = 90° and angle P′ the difference of longitude = 112° 58′ 45″: to find MS.

(Rule XVI) $\cos.\ \overset{-}{MS} = \overset{+}{\sin.}\ P'M\ \cos.\ \overset{-}{P'}$ (putting over P′M and P′ their proper signs, we find cos. MS is negative, p. 33, P. I.) ∴ distance MS = 112° 53′ 30′.

PROB. 113. (fig. 85.)

Let Z and S be the places of the ship, ZS = 100′ = 1° 40′ the arc described by ship. Draw meridians PZQ, PSQ,, then QZS will represent the course (=45°) on which she started, and PSZ or its opposite angle, the course of ship at point S. To find this angle, we have in triangle PZS, the two sides *PZ* the colat, and ZS the dist. and included angle PZS supplement of course: to find angle PSZ = 43° 38′ (Rule XIII).

PROB. 114. (figs. 86, 87).

Let A B C be the three places; then we have given PA = 50°, PB = 40°, AB = AC = BC = 20°: to find CP the colat. of C and angle CPB, from which its longitude may be found.

(1) In triangle ABP are given the three sides to find APB (= 24° 43′) and ABP (= 110° 31′ 15″).

(2) In triangle CBA the three sides are given to find CBA (= 61° 1′ 20″)

$$\therefore \text{CBP} = (\text{ABP} - \text{CBA}) = 49^\circ\ 29'\ 55''.$$

(3) In triangle CPB two sides CB (= 20°), PB (= 40°) and angle CBP (= 49° 29′ 55″) are given to find colat. CP (= 30° 23′) ∴ lat. = 59° 37′ N.

(4) In triangle CPB, are given three sides, to find angle CPB = 30° 56′ 15″, whence CPA (= CPB − APB) = 6° 13′ 15″.

If C is to the East of A (as in fig. 84) then longitude of C = long. A + 6° 13′ 15″ = 21° 13′ 15″ E.

If C is to the West of A (fig. 85), then long of C = long. A − 6° 13′ 15″ = 8° 46′ 45″ E.

PROB. 115. (fig. 88).

Let T and R represent the places of the stars, and C that of the comet: A the first point of Aries, AEQ the ecliptic, and $P_{/}$ its pole: draw circles of latitude $P_{/}T$, $P_{/}C$, and $P_{/}R$; then CE is the latitude and AE the longitude of the comet, to be found.

(1) In triangle $TP_{,}R$ .... $P_{,}T = 95° 28' 45''$, $P_{,}R = 89° 32' 30''$ (the ecliptic polar distances of the stars) and included angle $TP_{,}R = 80° 3' 15''$ (the difference of their longitudes; whence $TR = 80° 8' 45''$.

(2) In triangle $TP_{,}R$, the three sides are given to find angle $P_{,}TR = 88° 34' 45''$.

(3) In triangle TCR, the three sides are given to find angle $RTC = 39° 32' 15''$

$$\therefore P_{,}TC = 88° 34' 45'' + 39° 32' 15'' = 128° 7'.$$

(4) In triangle $P_{,}TC$, given $TP_{,} = 95° 28' 45''$, $TC = 40° 12'$ and included angle $P_{,}TC = 128° 7'$, to find $P_{,}C = 118° 0' 15''$

hence latitude CE of comet $= 28° 0' 15''$ S.

(5) In triangle $TP_{,}C$, given the three sides to find the angle $TP_{,}C = 35° 6' 40''$

hence long. AE of comet $=$ long. of T $+ 35° 6' 40''$ $= 67° 12' 15'' + 35° 6' 40'' = 102° 18' 55''$.

Prob. 116. (fig. 83).

Let S be the place of sun $18°$ below the horizon: then in the triangle ZPS are given polar dist. $PS = 81° 30'$, colat. $ZP = 35° 24'$ and zenith distance $ZS = 108°$, to find hour angle $ZPS = 9^h\ 14^m\ 16^s$.

Prob. 117. (fig. 89).

(Solution 1). Let X and Y be the places of the sun at the times of the observations. PX, PY the polar dis-

tances each $= 66° 34'$ and ZX, ZY the zenith distances $= 50° 10'$ and $61° 20'$, and angle XPY $= 1^h 30^m$ (the interval between the observations). It is required to find PZ the colatitude and thence the latitude. Draw XY an arc of a great circle passing through X and Y.

(1) In triangle PXY, given the polar distances PX, PY each $= 66° 34'$ and included angle XPY $= 1^h 30^m$; to find third side XY $= 20° 37' 21''$.

(2) In triangle PXY, given the three sides to find angle PYX $= 85° 28' 30''$.

(3) In triangle ZYX, given the three sides, to find angle ZYX $= 51° 41'$.

(4) Hence PYZ ($=$ PYX $-$ ZYX) $= 33° 47' 30''$.

Lastly. In triangle PYZ, are given PY, ZY and included angle PYZ, to find the third side PZ $= 30° 42'$ the colat. ∴ latitude $= 90° - 30° 42' = 59° 18'$ N.

2nd Sol. (fig. 90). Let X and Y be the places of the sun at the times of observation: PX PY the polar distances (which in this solution is always supposed to be equal). Bisect XY in M and join PM, ZM: then PM is at right angles to XY and ∴ the angle PMZ is the complement of ZMX: draw ZE perpendicular to PM.

Let a_1 a_2 represent the altitudes at X and Y
h the half interval MPY between observations.
d the decl. and l the latitude.

Let MX or MY $= x$; PM $= y$,
ZE $= u$, PE $= v$ ∴ ME $= y - v$.
ZM $= z$ and angle ZMP $= \theta$.

We have to compute the following arcs:

(1) x; (2) y; (3) (u); (4) $y-v$: thence subtracting arc. (4) from arc. (2) v is known:* and lastly, knowing u and v in the triangle PZE, PZ the colat. can be found.

In triangle PMY.... $\sin x = \sin. h \cos. d$...... (1)

$\sin. d = \cos. y \cos. x \therefore \cos. y = \sin. d \sec. x$...... (2)

In triangle ZME.... $\sin. u = \sin. \theta \sin. z$...... ($\alpha$)

$\cos. z = \cos. (y-v) \cos. u \therefore \cos. (y-v) = \cos z \sec. u$ (β)

To eliminate $\sin. \theta$ and $\cos. z$ from (α) and (β) we have in the triangle ZMX and XMY.

$$\cos. ZMX, \text{ or } \sin. \theta = \frac{\sin. a_1 - \cos. z \cos. x}{\sin. z \sin. x}$$

$$\cos. ZMY \text{ or} -\sin. \theta = \frac{\sin. a_2 - \cos. z \cos. x}{\sin. z \sin. x}$$

adding.... $0 = \sin. a_1 + \sin. a_2 - 2 \cos. z \cos. x$

$\therefore 2 \cos. z \cos x = 2 \sin. \frac{1}{2} (a_1 + a_2) \cos. \frac{1}{2} (a_1 - a_2)$

or $\cos. z = \sin. \frac{1}{2} (a_1 + a_2) \cos. \frac{1}{2} (a_1 - a_2) \sec. x$

Subtracting.... $2 \sin. \theta = \dfrac{\sin. a_1 - \sin. a_2}{\sin. z \sin. x}$

$\therefore \sin. \theta = \cos \frac{1}{2} (a_1 + a_2) \sin. \frac{1}{2} (a_1 - a_2) \operatorname{cosec} z \operatorname{cosec} x$

* If the great circle drawn through X and Y, pass when produced between P and Z the perpendicular ZE will fall without the triangle PZM; in this case ME $= v - y$ and v is found by *adding* arcs. (2) and (4) together: or since PE must evidently be less than 90° and the colatitude PZ is always less than 90° (unless the latitude is nothing): consequently when *MP + ME is equal to or greater than* 90°, PE or v cannot be equal to MP + ME: hence v = the *difference* MP—ME; or the difference of arcs. (2) and (4) must in such case be taken.

Substituting these values of sin. θ and cos. z in (β) and (α) we have

$$\sin. u = \cos. \tfrac{1}{2}(a_1 + a_2) \sin. \tfrac{1}{2}(a_1 - a_2) \operatorname{cosec} x \ldots. (3)$$

$$\cos(y - v) = \sin. \tfrac{1}{2}(a_1 + a_2) \cos. \tfrac{1}{2}(a_1 - a_2) \sec. x \sec. u \; (4)$$

Lastly, in the right angled triangle PZE,

$$\cos. PZ \text{ or } \sin. l = \cos. u \cos. v.$$

From this solution is deduced the Rule in Riddle's Navigation, for finding the latitude by double altitude; observing that in finding y we must take the supplement of the arc given by the tables when the lat. and declination are of different names.

From Solution 1, is derived the Rule for finding the latitude by double altitude given by Dr. Inman. This method is perfectly general since it may be applied to the same or different heavenly bodies, taken at the same or different times. The Rule obtained from the 2nd Solution is considered more concise when the same heavenly body is observed, if its declination does not alter in the interval between the observations, as when two altitudes of a star are taken. It will also give a near approximate latitude from two altitudes of the sun: the declination in this case being supposed to remain invariable and to be taken out of the Nautical Almanac for the middle time between the observations.

Prob. 118. (fig. 91).

Proceeding in a similar manner as in last Prob. (Solution 1), we find XY $= 22° \; 42'$; PYX $= 155° \; 38'$; ZYX$=89° \; 57'$ $\therefore$ PYZ$=65° \; 41'$. PZ$=63° \; 36' \; 53''$, whence latitude $= 26° \; 23' \; 7''$ N.

PROB. 119. (fig. 92).

Let A and B be the two places in the same latitude: PA, PB, their meridians; draw PD at right angles to AB the great circle passing through the two places; then PD bisects the angle APB since AP = BP: and D is the highest point reached by the ship sailing on a great circle from A to B. Produce also the meridians to the equator UV and draw the parallel of latitude $AD_{,}B$.

(1) In right angled triangle APD, given PA = 56° 9′, and angle APD = half difference of longitude between A and B = 68° 5′: to find AD = 50° 23′ 45″ and PD = 29° 6′. ∴ lat. of D = 60° 54′, and AB = 2.AD = 100° 47′ 45″ or 6047·5 nautical miles; the distance between A and B on a great circle.

(2) To find the distance $AD_{,}B$ on a parallel of latitude we have (by Trigonometry) $AD_{,}B$ = UV cos. AU = diff. long. × cos. lat.

= 8170 cos. 33° 51′ = 6785 miles

whence the difference on the two circles = 737·5 nautical miles.

PROB. 120. (fig. 93).

Let A and B represent the two places PA and PB their meridians: draw the perpendicular PD then the latitude of D is the highest attained by ship.

(1) In triangle APB, given PA=56° 9′, PB=34° 2′ and angle APB = 140° 27′: to find the angle PAB = 20° 59′ 43″ (Rule XIII).

(2) In right angled triangle PAD, given PA $= 56° 9'$ and angle PAB $= 20° 59' 45''$: to find PD $= 17° 18' 45''$ $\therefore$ lat. of D $= 72° 41' 15''$

Prob. 121.

This Prob. is already given (see 111).

Prob. 122. (fig. 94).

Let X and Y represent the two stars on the same circle of altitude. It is required to find the azimuth of X and Y; or the angle PZX.

(1) In triangle PXY, given PX $= 73° 49'$, PY $= 61° 35' 34''$ and included angle XPY $= 3^h 8^m 27^s$ (the difference of right ascensions); to find angle PXZ $= 65° 46' 15''$.

(2) In triangle PXZ, given PX, PZ, the colat. and opposite angle PXZ, to find opposite angle PZX $= 104° 55'$: whence its supplement $=$ S $75° 5'$ W. the bearing required.

Prob. 123. (fig. 95).

Let M and X represent the two bodies on the same vertical circle ZM: it is required to find PZ the colat. of place, and thence the latitude.

(1) In triangle PXM, given PX $= 77° 14' 15''$, PM $= 91° 42' 30''$ and included angle XPM $= 2^h 36^m 12^s$ (the difference of right ascensions); to find angle PMZ $= 68° 23' 15''$.

(2) In triangle PMZ, given PM and ZM the moon's zen. dist., and included angle PMZ, to find colat. PZ$=70^\circ\ 4'\ 22''$ and $\therefore$ lat.$=19^\circ\ 55'\ 38''$ N.

PROB. 124.

See Prob. 99.

PROB. 125. (fig. *b*).

Let D be the place of the heavenly body when setting, PD a circle of declination, and ZD a circle of altitude passing through D: then PZD is a quadrantal triangle, and we have given PD the polar distance $= 90^\circ - d$: PZ the colat. $= 90^\circ - l$: to find the hour angle ZPD or h, and PZD $= 90^\circ$ — amplitude, or $90^\circ - m$.

By form. (1), (p. 50) .... $\cos. h = -\tan. l \tan. d$.

$\therefore \cos. \overset{-}{h} = -\tan. \overset{+}{48^\circ} \tan. \overset{+}{20^\circ}$. $\therefore h = 7^h\ 35^m\ 22^s$ (placing the proper signs over the known quantities we find $\cos. h$ is negative or h greater than 6^h or 90°, see P. I. p. 32).

By form. (2) (p. 50).... $\sin. d = \cos. l \sin. m$.

$\therefore \sin. m = \sin. d \sec. l \quad \therefore \quad m =$ W $30^\circ\ 44'\ 30''$ N.

From this problem is derived the Rule in Navigation for finding the Amplitude of a heavenly body, or its true bearing when in the horizon.

PROB. 126. (fig. *b*).

Let Y and V represent the places of the sun when due West and at 6 o'clock,

$a =$ D,V the alt at. 6 o'clock

$a_{,} =$ WY alt. when on prime vertical.

$\therefore$ ZV $= 90 - a$, and ZY $= 90 - a_{,}$

Let $x =$ lat. and $y =$ decl. then PZ $= 90° - x$ and PV $=$ PY $= 90° - y$.

(By Rule XV. or formulæ and 5, p. 50, 51).

In tri. ZPV.. cos. $(90° - a) =$ cos. $(90° - y)$ cos. $(90 - x)$

$\therefore$ sin. $a =$ sin. y sin. x .... (1)

In tri. PZY.. cos. $(90° - y) =$ cos $(90° - x)$ cos. $(90° - a_{,})$

$\therefore$ sin. $y =$ sin. x sin. $a_{,}$ .... (2)

Multiplying (1) and (2) together and cancelling, we have.. sin. $a =$ sin.^{2} x sin. $a_{,}$ or sin.^{2} $x =$ sin. a cosec $a_{,}$ which formula determines x : and dividing (1) by (2), we have sin.^{2} $y =$ sin. a sin. $a_{,}$, whence y may be found.

PROB 127. (fig. b).

By form. (2) and (3) p. 50, or by Rules XVI, XV.

sin. $d =$ cos. l sin. m...... (1)

sin. $a =$ sin. d sin. l ...... (2)

in which formulæ a and m are given to find l and d: or since l and d are unknown let us denote them by x and y: then sin. $y =$ cos. x sin. m

and sin. $a =$ sin. y sin. x

multiplying them together and cancelling,

sin. $a =$ sin. x cos. x sin. m

or 2 sin. $a = 2$ sin. x cos. x sin. $m =$ sin. $2x$ sin. m

$\therefore$ sin. $2x = 2$ sin. a cosec m whence $2x = 84°$ or $96°$ (its supplement). $\therefore$ $x = 42°$ or $48'$. By substituting in (1) the values of x we have $d = 22°\ 20'$ or $20°$.

PROB. 128. (fig. b).

By formulæ (1) and (3) p. 50 or by Rules XVI, XV.

$$\cos. h = -\tan. y \tan. x \ldots\ldots (1)$$

$$\sin. a = \sin. y \sin. x \ldots\ldots\ldots (2)$$

x denoting the latitude, and y the declination.

$$\text{From (1)} \cos. h = -\frac{\sin. y \sin. x}{\cos. y \cos. x} = -\frac{\sin. a}{\cos. y \cos. x} . \text{(by 2)}$$

$$\therefore \cos. y \cos. x \cos. h = -\sin. a$$

$$\text{and } \cos. y \cos. x = -\sin. a \sec. h$$

$$\text{but by (2)..} \sin. y \sin. x = \sin. a$$

adding and subtracting, we have

$$\cos. x \cos. y + \sin. x \sin. y = \sin. a\,(1 - \sec. h)$$

$$\cos. x \cos. y - \sin. x \sin. y = -\sin. a\,(1 + \sec. h)$$

$$\text{or } \cos. (x - y) = \sin. a (1 - \sec. h) \ldots. (3)$$

$$\cos. (x + y) = -\sin. a (1 + \sec. h) \ldots. (4)$$

To reduce (3) and (4) to a logarithmic form.

$$\cos. (x - y) = \sin. a \left(1 - \frac{1}{\cos. h}\right) = \frac{\sin. a\,(\cos. h - 1)}{\cos. h}$$

$$= -2 \sin. a \sin.^2 \frac{h}{2} \sec. h \ldots\ldots\ldots\ldots (5)$$

$$\cos. (x + y) = -\sin. a \left(1 + \frac{1}{\cos. h}\right) = -\frac{\sin. a\,(\cos. h + 1)}{\cos. h}$$

$$= -2 \sin a \cos.^2 \frac{h}{2} \sec. h \ldots\ldots\ldots\ldots (6)$$

From equations (5) and (6) the values of $x-y$ and $x+y$ may be determined and thence x and y can be found.

PROB. 129. (fig. b).

By formula (1) p. 50, $\cos. h = -\tan. x \tan. y$ .... (1)
.... (6) p. 51, $\cos. h_{,} = \cot. x \tan. y$ ...... (2)

Dividing (1) by (2), $\dfrac{\cos. h}{\cos. h_{,}} = -\dfrac{\tan. x}{\cot. x} = -\tan^2 x$

or $\tan^2 x = -\cos. h \sec. h_{,}$: which equation determines the value of x; and multiplying (1) and (2) together, we have $\tan.^2 y = \cos. h \cos. h_{,}$ whence the declination can be found.

PROB. 130. (fig. b).

Let h = hour angle at setting, $h_{,}$ = hour angle when West, t=interval, $\therefore t = h - h_{,}$, d=decl. x=latitude.

$\cos. t = \cos. (h - h_{,}) = \cos. h \cos. h_{,} + \sin. h \sin. h_{,}$ .. (1)
by formula (1) p. 50 .. $\cos. h = -\tan. x \tan. d$ .. (2)
.... (6) p. 51 .. $\cos. h_{,} = \cot. x \tan. d$.... (3)
Multiplying (2) and (3) .... $\cos. h \cos. h_{,} = -\tan.^2 d$
or $2 \cos. h \cos. h_{,} = -2 \tan.^2 d$

Subtracting equation (1) from this equation

$$\cos. h \cos. h_{,} - \sin. h \sin. h_{,} = -\cos. t - 2 \tan.^2 d$$

$$\therefore -\cos. (h + h_{,}) = \cos. t + 2 \tan.^2 d.$$

This equation determines the value of $h + h_{,}$ the sum of the hour angles: having this, and the difference, we can find the angles h and $h_{,}$ and thence the latitude x. The operation is as follows.

To find value of $-\cos.(h+h_{,}) = \cos. t + 2\tan.^2 d$.
tab. log. cos. $t = 9{\cdot}864304$ $\therefore$ log. cos. $t = \bar{1}{\cdot}864304$
$\therefore$ cos. $t = {\cdot}73165$. To find value of $2 \tan^2 d$
log. $(2 \tan.^2 d) = \log. 2 + 2 \log. \tan. d - 20 = \bar{1}{\cdot}423162$
$\therefore$ $2 \tan.^2 d = {\cdot}26495$
$-\cos.(h + h_{,}) = {\cdot}73165 + {\cdot}26495 = {\cdot}9966$, angle corresponding to log. of this in tables $= 0^h\ 18^m\ 54^s$. Now since the $\cos.(h+h_{,})$ is *negative* the value of $h+h_{,}$ must be either $12^h - 18^m\ 54^s$ or $12^h + 18^m\ 54^s$.

that is	$h+h_{,} = 11^h$	41^m	6^s	or	12^h	18^m	54^s
and since	$h-h_{,} = 2$	51	54	..	2	51	54
	$\therefore$ $2h = 14$	33	0	or	15	10	48
	and.. $h = 7$	16	30	..	7	35	24

With the first value of h the lat. $= 42°$
...... other.............. $= 48$ nearly.

PROB. 131. (fig. *b*).

By form. 2, p. 50.... $\sin. d = \cos. l \sin. m$
...... 4, ... $\cos. l = \cot. d \cot. Z_{,}$
Multiplying .. $\sin. d \cos. l = \cos. l \sin. m \cot. d \cot. Z_{,}$

$$\therefore \sin.^2 d = \sin. m \cot. Z_{,} \cos. d$$

$$\text{or } 1 - \cos.^2 d = \sin. m \cot. Z_{,} \cos. d.$$

Whence $\cos.^2 d + \sin. m \cot. Z_{,} \cos. d = 1$
completing square, and extracting the root,

$$\cos. d = -\tfrac{1}{2} \sin m \cot. Z_{,} \pm \tfrac{1}{2}\sqrt{\sin.^2 m \cot.^2 Z_{,} + 4}$$

$$= -0{\cdot}06224 + 1{\cdot}00193 = {\cdot}93969$$

$\therefore$ tab. log. cos. $d = 9{\cdot}972984$ $\therefore$ $d = 20°$
and thence l may be found.

PROB. 132. (fig. b).

Let a=altitude at 6 o'clock.
m=meridian altitude.
d and l the decl. and lat.

Then since lat. (ZQ)=mer. zen. dist. (Zm)+decl.(mQ)

(See construction of fig. to Prob. 88)

or $l=90°-m+d \quad \therefore \ d=l+m-90°$.

$\therefore$ sin. d=sin. $(l+m-90°)$ = $-$sin. $(90°-\overline{l+m})$
$= -$ cos. $(l+m)$.

But sin. a=sin. l sin. d = $-$sin. l cos. $(l+m)$
or 2 sin. l cos. $(l+m) = -2$ sin. a.
$\therefore$ sin. $(2l+m)-$sin. $m=-2$ sin. a.
or sin. $(2l+m)$ = sin. m $-$ 2 sin. a.

Calculation.

log. sin. m .... $\bar{1}\cdot945935$ $\quad\therefore$ nat. sine .... 0·88295
.. sin. a .... $\bar{1}\cdot405141$ $\quad\therefore$ nat. sine .... 0·25418

$\therefore$ sin. $m-2$ sin. a=0·37459=nat. sin. $(2l+m)$.
$\therefore$ tab. log. sin. $(2l+m)$=9·573556
$\therefore$ $2l+m=158°$ (suppl. of 22°)
whence $l=48°$ N.

PROB. 133. (fig. b).

Let h = hour angle at setting
$h_{\prime}$= .... when west.

By form. (1) ...... cos. h = $-$ tan. l tan. d
and (6).......... cos. $h_{\prime}$= cot. l tan. d

adding and .... $\cos. h + \cos. h_{,} = (\cot. l - \tan. l) \tan. d$

subtracting .. $\cos. h - \cos. h_{,} = -(\tan. l + \cot. l) \tan. d$

dividing...... $\dfrac{\cos. h + \cos. h_{,}}{\cos. h - \cos. h_{,}} = -\dfrac{\cot. l - \tan. l}{\tan. l + \cot. l}$

$$\text{or } \frac{\cos. h + \cos. h_{,}}{-(\cos. h - \cos. h_{,})} = \frac{\cot. l - \tan. l}{\tan. l + \cot. l}$$

$$\therefore \frac{2 \cos. \frac{1}{2}(h + h_{,}) \cos. \frac{1}{2}(h - h_{,})}{2 \sin. \frac{1}{2}(h + h_{,}) \sin. \frac{1}{2}(h - h_{,})} = \frac{1 - \tan.^2 l}{\tan.^2 l + 1} = \cos. 2l$$

$$\text{or } \cot. \tfrac{1}{2}(h + h_{,}) . \cot. \tfrac{1}{2}(h - h_{,}) = \cos. 2\,l$$

$$\therefore \cot. \tfrac{1}{2}(h + h_{,}) = \cos. 2\,l \tan. \tfrac{1}{2}(h - h_{,})$$

which equation determines $\frac{1}{2}(h + h_{,})$ and, with $\frac{1}{2}(h - h_{,})$ already known, the angles h and $h_{,}$ may be found, and thence by form. (1) we can find d.

Prob. 134. (fig. 94).

Suppose the heavenly body to rise at S and to describe the parallel of declination SX. Let a circle of altitude ZO touch the parallel SX at X: then X is the point where the azimuth is the greatest: the azimuth increasing from the point S, where the heavenly body rose, to X and beginning to decrease after passing X. The angle PXZ is evidently a right angle: whence the required azimuth, hour and altitude may all be found from the right angled triangle PZX.

(1) To find greatest azimuth PZX

$\cos. d = \cos. l \sin. az. \quad \therefore \sin. az. = \cos. d. \sec. l.$

(2) To find ZPX.... cos. h = cot. d tan. l.

(3) To find altitude XO or a,

sin. l = sin d sin a $\therefore$ sin a = sin l cosec d.

Prob. 135. (fig. 94).

The preceding problem furnishes an explanation of the fact, that when the sun's declination is greater than the latitude of a place, and of the same name with it, the shadow of an upright object on a horizontal plane goes backward each day during a certain period, which may be computed, as well as the arc through which it goes back.

At the point X the sun appears stationary in azimuth.

PZS is its bearing when rising.

PZO............ at point X

The angle SZO or arc SO will therefore denote the number of degrees the bearing has increased from sun rising, or that the shadow of the object has gone back; and the period during which the retrograde motion has taken place will be measured by the angle SPX.

(1) To find bearing PZS at rising.

In tri. PZS.... sin. d = cos. l cos. PZS

whence PZS.... = 64° 55′ 30″

By last prob.... PZX.. = 77 28 0

$\therefore$ Shadow goes back .... = 12 32 30

(2) To find hour angle ZPS at rising

$$\text{In tri. PZS.. cos. } \overset{-}{\text{ZPS}} = -\text{tan.} \overset{+}{l} \text{ tan.} \overset{+}{d}$$

$$\begin{array}{lrrr} & \text{h} & \text{m} & \text{s} \\ \therefore \text{ZPS} \therefore = & 6 & 36 & 22 \\ \text{By last prob...... ZPX .. =} & 2 & 12 & 7 \\ \hline \therefore \text{ time of shadow's going back.. =} & 4 & 24 & 15 \end{array}$$

PROB. 136. (fig. 97).

(1st Sol.) Let S and M be the respective places of the sun and moon at the time of observation; d and $-d$ the decl. of sun and moon respectively: that of moon being taken negatively, since the bodies are on different sides of the equator h and $h_{\prime}$ their hour angles.

then $h - h_{\prime} = 1^{h}\ 53^{m}\ 42^{s}$.

In triangle ZPS.... cos. $h = -$tan. l tan. d........ (1)

In triangle ZPM .. cos. $h_{\prime} = -$tan. l tan. $(-d_{\prime})$

$= \text{tan. } l \text{ tan. } d_{\prime}$........ (2)

dividing (2) by (1) .. $\dfrac{\text{cos. } h_{\prime}}{\text{cos. } h} = \dfrac{\text{tan } d_{\prime}}{-\text{tan. } d}$

$$\text{or } \frac{\text{cos. } h_{\prime} + \text{cos. } h}{\text{cos. } h_{\prime} - \text{cos. } h} = \frac{\text{tan. } d_{\prime} - \text{tan. } d}{\text{tan. } d_{\prime} + \text{tan. } d}.$$

$$\therefore \frac{2\text{cos.}\frac{1}{2}(h_{\prime}+h)\text{cos.}\frac{1}{2}(h_{\prime}-h)}{-2\text{sin.}\frac{1}{2}(h_{\prime}+h)\text{sin.}\frac{1}{2}(h_{\prime}-h)} = \frac{\text{sin.}d_{\prime}\text{cos}d - \text{cos } d_{\prime}\text{sin.}d}{\text{sin.}d_{\prime}\text{cos}d + \text{cos.}d_{\prime}\text{sin.}d}$$

$$\text{or} -\text{cot. } \tfrac{1}{2}\,(h_{\prime}+h) \text{ cot. } \tfrac{1}{2}\,(h_{\prime}-h) = \frac{\text{sin.}(d_{\prime}-d)}{\text{sin.}(d_{\prime}+d)}$$

$\therefore \cot.\frac{1}{2}(h_{,}+h) = -\sin.(d_{,}-d)\operatorname{cosec}(d_{,}+d)\tan.\frac{1}{2}(h_{,}-h)$

$= \sin.\ (d_{,}-d)\ \operatorname{cosec}\ (d_{,}+d)\ \tan.\frac{1}{2}(h-h_{,})$

whence $\frac{1}{2}\ (h_{,}+h) = 5^{h}\ 53^{m}\ 29^{s}$

and since $\frac{1}{2}\ (h-h_{,}) = \quad 56 \quad 51$

$\therefore h \ldots = 6 \quad 50 \quad 20$

hence app. time $= 12^{h} - 6^{h}\ 50^{m}\ 20^{s} = 5^{h}\ 9^{m}\ 40^{s}$ A. M.

To find lat. l.. $\overset{-}{\cos.}\ h = -\ \overset{+}{\tan.}\ l.\ \overset{+}{\tan.}\ d$

or $\tan.\ l = \cos.\ h \cot.\ d \quad \therefore l = 50^{\circ}\ 18'\ 20''$ N.

(2nd Sol.) more easily thus:

(1) In tri. PMS, given PS, PM and included angle SPM to find angle PSM $= 128^{\circ}\ 33'\ 30''$.

(2) In right angled triangle NPS, given PS and angle PSN (supplement of PSM) to find lat. NP $= 50^{\circ}\ 18'\ 20''$ N.

Prob. 137.

Let h and $h_{,}$ denote the hour angles of the sun at rising or setting at the two places: then since it rises and sets at the same instant of absolute time at the two places the difference of the hour angles h and $h_{,}$ or $h-h_{,}$ must be equal to the difference in their reckoning of time, or to their difference of longitude (in time) $= 28^{\circ}\ 46'$.

Let l and $-\ l_{,}$ denote the latitudes of Dublin and Pernambuco: the latitudes being affected with different signs since the places are on opposite sides of the equator.

then $\cos. h = -\tan. l \tan. d$.............. (1)

$\cos. h_{,} = -\tan\ (-l_{,}) \tan. d = \tan. l_{,} \tan. d$ .. (2)

$$\therefore \frac{\cos. h_{,}}{\cos. h} = \frac{\tan. l}{-\tan. l} \text{ or } \frac{\cos. h_{,} + \cos. h}{\cos. h_{,} - \cos. h} = \frac{\tan. l_{,} - \tan. l}{\tan. l_{,} + \tan. l}$$

$$\therefore \frac{\cos. \frac{1}{2}(h_{,}+h) \cos. \frac{1}{2}(h_{,}-h)}{-\sin. \frac{1}{2}(h_{,}+h) \sin. \frac{1}{2}(h_{,}-h)} = \frac{\sin. (l_{,}-l)}{\sin. (l_{,}+l)} = \frac{-\sin. (l-l_{,})}{\sin. (l+l_{,})}$$

or $\cot. \frac{1}{2}(h_{,}+h) = -\sin. (l-l_{,}) \operatorname{cosec}. (l+l_{,}) \tan. \frac{1}{2}(h-h_{,})$. This equation determines half the sum of the hour angles $= 101° 40' 30''$, and thence with $h-h_{,}$ already known, the values of h or $h_{,}$ and by (1) the declination $d = 18° 6'$. Then the days on which the declination of the sun is the value of d thus found will be those required.

PROB. 138. (fig. 98).

Let ZQH represent the celestial meridian, H the horizon, Z the zenith, Q the equator, $S_{,}$ and S the places of the sun on the longest and shortest days.

Then $ZQ =$ lat. (x) and $QH = 90° - x$.

$SQ = S_{,}Q =$ decl. $(d) = 23° 28'$.

Let NG, the shadow of object PN when sun is at $S = y$; then NC its shadow when at $S_{,} = 7y$.

Now angle NPC or arc $ZS = x + d$

and angle NPG or arc $ZS_{,} = x - d$.

In triangle NPC.. $7x = PN \tan. NPC = PN \tan. (x+d)$

........NPG.. $x = PN \tan. NPG = PN \tan. (x-d)$

$$\text{dividing,.. } 7 = \frac{\tan. (x+d)}{\tan. (x-d)} = \frac{\sin. 2x + \sin. 2d}{\sin. 2x - \sin. 2d}$$

$$\therefore \frac{8}{6} \text{ or } \frac{4}{3} = \frac{\sin. 2x}{\sin. 2d} \text{ or } \sin.\ 2x = \frac{4}{3} \sin\ 2d.$$

whence $x = 38^\circ\ 27'\ 45''$.

Prob 139. (fig. 98).

Let $NG = x$, $NC = y$ $l, =$ lat. $d =$ decl.

then $x = PN \tan. (l-d)$; $y = PN \tan. (l+d)$

$$\therefore \frac{y}{x} = \frac{\tan. (l+d)}{\tan. (l-d)} = \frac{\sin. 2\,l + \sin. 2\,d}{\sin. 2\,l - \sin. 2\,d}$$

$$= \frac{\sin. 90^\circ + \sin. 30^\circ}{\sin. 90^\circ - \sin. 30^\circ} = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} = \frac{3}{1}$$

or $y : x :: 3 : 1$.

Prob. 140.

In fig. 98, if $S_{,}$ be the place of the sun and NC the shadow of object PN then $\frac{PN}{NC} = \frac{3}{5} = \tan. PCN =$ tan. sun's alt. $= .6$ $\therefore$ log. tan. alt. $= 9{\cdot}778151$ $\therefore$ alt. $= 30^\circ\ 58'$.

Then in triangle PZX (fig. 73), the three sides are given, viz. $PZ = 56^\circ\ 30'$, $PX = 79^\circ\ 45'$ and $ZX = 59^\circ\ 2'$, to compute the hour angle ZPX or h hence $h = 3^h\ 58^m\ 4^s$.

Prob. 141. (fig. 99).

Let AB represent the shadow of perpendicular stick AD, and AC its shadow when the stick is placed in the

position AT perpendicular to ray SC in which position it will manifestly throw the greatest shadow. Now the distance of S is so great compared with BC that the angle BSC has no assignable magnitude, and the rays SB, SC may therefore be considered parallel, or the angle ABD = angle ACT = sun's altitude required.

In triangle ABD .. AD = AB tan. alt. = 4 tan. alt.

In ATC .. AT = AD = AC sin. alt. = 5 sin. alt.

$\therefore$ 5 sin alt. = 4 tan. alt. = 4 sin. alt. sec. alt.

$$\therefore \text{sec. alt.} = \tfrac{5}{4} = 1\cdot25$$

$$\text{log. sec. alt.} = 10\cdot096910$$

$$\therefore \text{alt.} = 36^\circ\ 52'\ 15''$$

PROB. 142. (fig. 100).

Let A be the observer, B the shadow of cloud, C and D the place of sun. Since the distance AB will subtend no sensible angle at the sun the rays AD and BD may be considered parallel and angle CBD = DAB = sun's alt. = 22°, and the angle of elevation of cloud = 20°. $\therefore$ ACB = 2°.

$$\text{In tri. ABC} \;..\; \frac{BC}{AB} = \frac{\sin.\ 20^\circ}{\sin.\ 2^\circ} \;\therefore BC = \frac{AB \sin.\ 20^\circ}{\sin.\ 2^\circ}$$

In right angled triangle CBE.. CE = CB sin. 22°

$\therefore$ CE = AB sin. 20° sin. 22° cosec. 2° = 1468.

PROB. 143. (fig. *b*).

Let a = meridian alt. : h = hour angle at rising or setting, d = decl., l = lat.

$\therefore\ d = l -$ mer. zen, dist. $= l - \overline{90^\circ - a} = l + a - 90^\circ$.

In triangle PZD.. $\cos. h = -\tan. d \tan. l$.

$= -\tan. (l + a - 90^\circ) \tan. l = -\tan. -(90^\circ - \overline{l+a}) \tan. l$

$= \tan. (90^\circ - l + a) \tan. l = \cot. (l+a) \tan. l$

$$= \frac{\cos. (l+a) \sin. l}{\sin. (l+a) \cos. l}.$$

$$\therefore \frac{\cos. h + 1}{\cos. h - 1} = \frac{\cos. (l + a) \sin. l + \sin. (l + a) \cos. l}{\cos. (l + a) \sin. l - \sin. (l + a) \cos. l}$$

$$\text{or } \frac{2 \cos.^2 \frac{h}{2}}{-2 \sin.^2 \frac{h}{2}} = \frac{\sin. (2l + a)}{-\sin. a}$$

$$\therefore \cot.^2 \frac{h}{2} \sin. a = \sin. (2l + a)$$

whence $2l + a = 29^\circ\ 13'$ or $150^\circ\ 47'$ (its supplement) taking this last value $2l = 94^\circ\ 47'$ $\therefore\ l = 47^\circ\ 23'\ 30''$

To avoid the ambiguity arising from obtaining the value of $2l + a$ in terms of the sine; let $z = 90^\circ - a$ the zenith dist., then $d = l - z$.

$\therefore\ \cos. h = -\tan. d \tan. l = -\tan. (l - z) \tan. l$

$$= \frac{-2 \sin. (l - z) \sin. l}{2 \cos. (l - z) \cos. l} = \frac{\cos. (2l - z) - \cos. z}{\cos. (2l - z) + \cos. z}$$

$$\therefore \frac{\cos. h + 1}{\cos. h - 1} = \frac{\cos. (2l - z)}{-\cos. z}$$

$$\text{or } \cos. (2l - z) = \cos. z \cot.^2 \frac{h}{2}$$

from which equation the value of l may be obtained free from ambiguity.

PROB. 144. (fig. 101).

Let h denote the hour angle of the sun at rising, or half the length of the day: and since the meridian altitude is less than the latitude, the declination must be of a different name to latitude, or south.

In quadrantal triangle $PZ_{\prime}S$..
$\cos. h = -\tan. l \cot. (90^\circ + d) = \tan. l \tan. d = \tan. d,$
Since $\tan. l = \tan. 45^\circ = 1.$

$$\therefore \cos. h = \tan. (z - l) \quad \text{if } z = \text{zen. dist.}$$

$$= \frac{\tan. z - \tan. l}{1 + \tan. z \tan. l} = \frac{\tan. z - 1}{\tan. z + 1}$$

$$\therefore \frac{\cos. h + 1}{\cos. h - 1} = -\tan. z .. \text{ or } = \frac{1 + \cos. h}{1 - \cos. h} = \tan. z.$$

$$\therefore \cos.^2 \frac{h}{2} = \tan. z = \cot. \text{alt.} = \cot. 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \frac{h}{2} = \frac{1}{\sqrt[4]{3}}.$$

PROB. 145.

Let z = zenith dist. $\therefore z = 2l.$
and (fig. 66) $z = l + d \quad \therefore 2l = l + d$ or $l = d.$
In triangle $PZ_{\prime}S$ (fig. 101) $\cos. h = -\tan. l \cot. (90^\circ + d)$
$= \tan. l \tan. d = \tan.^2 l.$

$$\therefore \tan.^2 l = \cos. 7^h \text{ whence } l = 26^\circ\ 58'.$$

PROB. 146. (fig. *b*).

Complete triangle PZD_2. then $\cos. h = -\tan. l \tan. d$ $= -\tan. d$; since $l = 45°$; but $d = l - z$

$$\therefore \cos. h = -\tan. (l - z)$$

$$= -\frac{\tan. l - \tan. z}{1 + \tan. l \tan z} = \frac{\tan. z - 1}{\tan. z + 1}.$$

$$\text{or } \frac{\cos. h + 1}{\cos. h - 1} = -\tan. z = -\cot. \text{alt.}$$

$$\therefore \cot. \text{alt.} = \cot.^2 \frac{h}{2} = \cot.^2 60° = \frac{1}{3}$$

$$\therefore \tan. \text{alt.} = 3.$$

PROB. 147. (fig. 102).

Let $S_{,}$ be the apparent place of the sun when its upper limb touches the horizon : S its apparent place when lower limb touches horizon : then angle $S_{,}PS$ measures the time elapsed between the upper and lower limbs being in horizon.

$$S_{,}PS = ZPS_{,} - ZPS.$$

(1) In triangle $ZPS_{,}$ given $ZP = 40°$, $ZS_{,} = 90° \, 16'$ and $PS_{,} = 84° \, 22'$ $\therefore$ angle $ZPS_{,} = 6^h \, 28^m \, 41^s$.

(2) In triangle ZPS given $ZP = 40°$, $ZS = 90° - 16'$ $= 89° \, 44'$ and $PS = 84° \, 22'$ $\therefore$ $ZPS = 6^h \, 25^m \, 22^s$

$$\therefore \text{angle } S_{,}PS = 3^m \, 19^s.$$

PROB. 148. (fig. 103).

Let AB represent the arc of the parallel of decl. described by sun's center: P the pole of the heavens: then considering the triangle APB as a spherical triangle, (which may be done in this case since arc AB being small does not differ much from the corresponding arc of a great circle): we have the three sides of APB given, to find the angle P, the time of semidiameter passing meridian. Now by the common formula;

$$\text{Sin.}^2 \frac{P}{2} = \text{cosec. AP cosec PB sin. } \tfrac{1}{2} \text{ (AB} + \text{AP} - \text{PB)}$$

$$\text{sin. } \tfrac{1}{2} \text{ (AB} - \overline{\text{AP} - \text{PB}}) = \text{cosec.}^2 \text{AP sin.}^2 \tfrac{1}{2} \text{ AB}$$

$$\therefore \text{ sin. } \frac{P}{2} = \text{sec. } d \text{ sin. } 8' \, 8''.6$$

$$\text{and P} = 1^m \; 11^s \text{ nearly.}$$

Or thus: Let AB represent an arc of a small circle equal in length to the sun's semidiameter. A the sun's center: through A and B draw circles of declination PU, PV, cutting the equator in U and V. Then the time of the point A describing the arc AB will be measured by the arc UV or the angle UPV.

Now by Trig. UV = AB sec. AU = $16' \, 17''.3$ sec. d

or arc. UV (in time) $= \dfrac{977 \cdot 3}{15}$ sec. $d = 1^m \; 10^s.82$.

(this is independent of the sun's apparent motion from west to east.)

PROB. 149. (fig. 104).

Let A be the first station of ship; B the second, and C the point of land.

(1) In triangle ABC given AB = 6 miles, angle CAB, (the difference of bearing between C and B) = 2 points, and angle ACB = $4\frac{1}{2}$ points : to find AC.

(2) Draw AD due east and CD at right angles to it: then CD = difference of latitude between A and C and (considering ADC as a right angled plane triangle) CD = AC sin. CAD = AC sin. 4 points = $5'\ 15''$ ∴ Lat. of C = $59°\ 6' + 5'\ 15'' = 59°\ 11'\ 15''$, also (the triangle ADC being isosceles) AD = $5'\ 15''$. To find the difference of longitude between A and C we have (by trig.) AD = diff. long. × cos. lat. A, whence diff. long. = $10'\ 13''$ E.

∴ Long. of C = $6°\ 15' + 10'\ 13'' = 6°\ 25'\ 13''$ E.

PROB 150. (fig 73).

Let Z be the zenith, P the pole and X the place of the heavenly body: then in the spherical triangle ZPX are given ZP = colat. (= $39°\ 12'$), PX = polar distance (= $77°\ 31'$) and included angle ZPX the hour angle (= $2^h\ 53^m\ 1^s$) to find the angle PZX, the true bearing or azimuth.

Calculation. (Rule XIII).

Pol. dist. .. 77° 31′	cot. $\frac{1}{2}h$	.. 0·401830	cot $\frac{1}{2}h$	.. 0·401830
Colat .. 39 12	sec. S	.. 0·280167	cosec S	.. 0·069894
	cos. D	.. 9·975255	sin. D	.. 9·516112
sum 116 43				
diff. 38 19	tan.	.... 10·657252	tan.	.... 9·987836
$\frac{1}{2}$ sum (S) .. 58 21$\frac{1}{2}$	arc. (1)	.. 77° 35′	arc. (2)	.. 44° 12′
$\frac{1}{2}$ diff. (D) .. 19 9$\frac{1}{2}$	arc. (2)	.. 44 12		

$\frac{1}{2}$ Hour angle (h) $1^h\ 26^m\ 30^s$ N.. 121 47.. E the bearing required.

From this problem is deduced the Rule in Navigation for finding the true bearing of a heavenly body when the time at the ship or place of observation is given: the latitude of place and declination of body being supposed to be known.

Prob. 151. (fig. 97).

Let S and M be the places of sun rising on the longest and shortest days.

h = hour angle of sun at S.

$h_{,}$ = ... M.

then $2(h-h_{,})$ = difference in length of day.

In triangle ZPS .. $\cos. h = -\tan. d \tan. l$ .... (1)

.... ZPM .. $\cos. h_{,} = +\tan. d \tan. l$ .... (2)

$\therefore \cos. h = -\cos. h_{,} = \cos. (12^h - h_{,})$.

$\therefore h = 12 - h_{,}$

or $h - h_{,} = 12 - 2h_{,} = \frac{1}{2}$ difference required.

Now in (2) $\cos. h_{,} = \tan. d \tan. l = \tan. d$ since $l = 45°$

Calculation.

$$\log.\ \tan.\ d \ldots.\ 9\cdot637611 = \log.\ \cos.\ h_{,}.$$

$$\therefore\ h_{,} = 4^{h}\ 17^{m}\ 5^{s} \text{ and } 2h_{,} = 8^{h}\ 34^{m}\ 10^{s}.$$

$$\therefore\ h-h_{,}\ (= 12-2h_{,}) = 3^{h}\ 25^{m}\ 50^{s}$$

and difference in length of days $= 6 \quad 51 \quad 40.$

PROB. 152. (fig. 97).

Let $h =$ hour angle of the sun at S.

$h_{,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ M. $\therefore\ h = h_{,}+3.$

In triangle ZPS.. $\cos.\ (h_{,}+3) = -\tan.\ d \tan.\ l \ldots (1)$

$\ldots\ldots$ ZPM.. $\cos.\ h_{,} = \tan.\ d \tan.\ l \ldots (2)$

$$\therefore\ \cos.\ (h_{,}+3) = -\cos.\ h_{,} = \cos.\ (12-h_{,})$$

$$\therefore\ h_{,}+3 = 12-h_{,} \text{ or } h_{,} = 4^{h}\ 30^{m}.$$

whence by (2) $l = 46^{\circ}\ 46'.$

PROB. 153.

Let h and $h_{,}$ be the hour angle, then $h-h_{,} = 1^{h}.$

d and $d_{,}$ the decl. at these times, viz. 20° and 10°

then $\cos.\ h = -\tan.\ l \tan.\ d \ldots\ldots\ldots (1)$

$\cos.\ h_{,} = -\tan.\ l \tan.\ d_{,} \ldots\ldots (2)$

$$\therefore\ \frac{\cos.\ h}{\cos.\ h_{,}} = \frac{\tan.\ d}{\tan.\ d_{,}} \text{ or } \frac{\cos.\ h+\cos.\ h_{,}}{\cos.\ h-\cos.\ h_{,}} = \frac{\tan.\ d+\tan.\ d_{,}}{\tan.\ d-\tan.\ d_{,}}$$

$$\text{or } \frac{2\cos.\ \frac{1}{2}(h+h_{,}) \cos.\ \frac{1}{2}(h-h_{,})}{-2\sin.\ \frac{1}{2}(h+h_{,}) \sin.\ \frac{1}{2}(h-h_{,})} = \frac{\sin.\ (d+d_{,})}{\sin.\ (d-d_{,})}$$

$$\therefore\ -\cot.\ \frac{1}{2}(h+h_{,}) \cot.\ \frac{1}{2}(h-h_{,}) = \frac{\sin.\ 30^{\circ}}{\sin.\ 10}$$

$$\therefore -\overset{-}{\text{cot.}}\ \tfrac{1}{2}(h+h_{,}) = \overset{+}{\text{tan.}}\ 7^{\circ}\tfrac{1}{2}\ \overset{+}{\text{sin.}}\ 30^{\circ}\ \overset{+}{\text{cosec}}\ 10^{\circ}$$

$$\begin{aligned} \text{whence } \tfrac{1}{2}(h+h_{,}) \ldots\ldots &= 7^{h}\ 23^{m}\ 3^{s} \\ \text{and } \tfrac{1}{2}(h-h_{,}) \ldots\ldots &= 0\ \ 30\ \ 0 \\ \therefore h \ldots\ldots &= 7\ \ 53\ \ 3 \end{aligned}$$

with this value of h, the lat. (by 1) $= 52^{\circ}\ 27'$ N.

PROB. 154. (fig. 97).

Let h be the hour angle of sun at S

$h_{,} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ M $\therefore h = 3h_{,}$

then $\cos. h = -\tan. l \tan. d$ …… (1)

$\cos. h_{,} = +\tan. l \tan d$ …… (2)

$\therefore \cos. h = -\cos. h_{,} = \cos. (12^{h} - h_{,})$

$\therefore h = 12 - h_{,}$ or $3h_{,} = 12 - h_{,}$ $\therefore 4h_{,} = 12$

and $h_{,} = 3$ hours whence by (2) $l = 58^{\circ}\ 27'$ N.

PROB. 155.

Let h and $h_{,}$ be the hour angles,

$$\text{then } h_{,} = \frac{3}{5} h$$

$$\begin{aligned} \text{and } \cos. h &= -\tan. d \tan. l \\ \cos. h_{,} &= \tan. d \tan. l \\ \therefore \cos. h &= -\cos. h_{,} = \cos. (12 - h_{,}) \\ \therefore h &= 12 - h_{,} = 12 - \frac{3}{5} h \end{aligned}$$

$$\therefore \frac{8h}{5} = 12 \text{ and } h = \frac{60}{8} = 7^{h}\ 30^{m}$$

with this value of h the lat. $= 41^{\circ}\ 24'$ N.

Prob. 156.

Let h denote the hour angle at the two places when the altitudes of the sun are the same.

l=lat. of one place=50° $l_{,}$=lat. of the other=40

d=decl. and z=zenith distance

$$\text{then } \cos. h = \frac{\cos. z - \sin. d \sin. l}{\cos. d \cos. l}$$

$$\text{and } \cos. h = \frac{\cos. z - \sin. d \sin. l_{,}}{\cos. d \cos. l_{,}}$$

$$\therefore \frac{\cos. z - \sin. d \sin. l}{\cos. d \cos. l} = \frac{\cos. z - \sin. d \sin. l_{,}}{\cos. d \cos. l_{,}}$$

$$\text{or } \cos. z \cos. l_{,} - \sin. d \sin. l \cos. l_{,} = \cos. z \cos. l - \sin. d \sin. l_{,} \cos. l$$

$$\text{or } \cos. z\,(\cos. l_{,} - \cos. l) = \sin. d\,(\sin. l \cos. l_{,} - \cos. l \sin. l_{,})$$

$$= \sin. d \sin. (l - l_{,})$$

$$\therefore \cos. z = \frac{\sin. (l-l_{,})}{\cos. l_{,} - \cos. l} \cdot \sin. d.$$

$$= \frac{2\sin.\frac{1}{2}(l-l_{,})\cos.\frac{1}{2}(l-l_{,})}{2\sin.\frac{1}{2}(l+l_{,})\sin.\frac{1}{2}(l-l_{,})} \cdot \sin d.$$

$$= \cos. \tfrac{1}{2}\,(l-l_{,}) \operatorname{cosec} \tfrac{1}{2}\,(l+l_{,}) \sin. d.$$

From this formula z may be found=61° 41′ 15″. Then in the triangle ZPS the three sides are known, hence the hour angle h may be found = $4^{h}\ 51^{m}$.

Prob. 157. (fig. 105).

Let A be the place sailed from.

B . . . . arrived at.

the arc $AA_{,}$ = arc $A_{,}B = a$.

and $BC = b$.

EQ the arc of equator between meridians of A and $A_{,}$.

Let x = lat. of A; then $x-a$ = lat. of B.

$$\text{(By Trig.)}\ \frac{\text{CB}}{\text{EQ}} = \frac{b}{\text{EQ}} = \cos.\ \text{BQ} = \cos.\ (x-a).$$

$$\ldots\ldots\ \frac{\text{AA}_{\prime}}{\text{EQ}} = \frac{a}{\text{EQ}} = \cos.\ \text{AE} = \cos.\ x.$$

$$\therefore \frac{b}{\cos.(x-a)} = \frac{a}{\cos.\ x}.$$

$$\text{or } \frac{b}{a} = \frac{\cos.\ (x-a)}{\cos.\ x} = \frac{\cos.\ x \cos.\ a + \sin.\ x \sin.\ a}{\cos.\ x}$$

$$= \cos.\ a + \tan.\ x \sin.\ a.$$

$$\therefore \frac{\frac{b}{a} - \cos.\ a}{\sin.\ a} = \tan.\ x.$$

Calculation.

$b = 150$
$a = 100$

$$\tan.\ x = \frac{1 \cdot 5 - \cos.\ 1^\circ\ 40'}{\sin.\ 1^\circ\ 40'}$$

log. cos. $1^\circ\ 40'$.. $\bar{1}\cdot 999816$ $\therefore$ cos. $1^\circ\ 40'$.... $\cdot 9994$

$$\tan.\ x = \frac{1 \cdot 5 - \cdot 9994}{\sin.\ 1^\circ\ 40'} = \cdot 5006.\ \text{cosec}\ 1^\circ\ 40'$$

$\therefore$ x or lat. A.. $= 86^\circ\ 40'\ 30''$ N.
Difference of lat... = 1 40 0 S.

Lat. of B required.. = 85 0 30 N.

Prob. 158. (fig. 106).

Let XZ be the apparent zenith distance of sun
SZ .... true ....
O the object whose bearing is required.

First suppose O to be in the horizon.

(1) In quadrantal triangle ZOX there are given OX the observed distance of object from sun, ZX the observed zenith distance and OZ = 90°; to find the angle OZX.

(2) In the triangle PZS are given PZ the colat, PS the polar distance of sun, and ZS the true zenith distance; to find the true bearing of sun, or the angle PZX.

The sum or difference of the two angles thus found according to the position of the object with respect to the sun, will be the bearing PZO of object required.

Calculation.

1. To find angle OZX.

App. Alt. ☉'s L. L. .	10° 30′ 0″	ob. dist. N. L. .	95° 16′ 0″
index corr. . . .	0 50—	in. cor.	1 10+
	10 29 10		95 17 10
semi.	15 45	semi. .	15 45
	10 44 55	OX. . . .	95 32 55
dip.	3 41	In quadral. tri. ZXO	
App. alt. ☉'s center. .	10 41 14	cos. OX = sin. ZX cos. OZX	
cor. for refraction, &c.	4 50	— + —	
		or cos.OX sec. ap. alt. = cos.OZX	
Tr. alt. ☉'s center . .	10 36 24	cos. OX	8·985491
		sec. A alt.	0·007600
		180°	
		84 21. .	8·993091
		OZX = 95 39	

(2) To find PZS.				0·004089			
lat.....	7°	51′	0″	0·007487			
Tr. alt...	10	36	24	4·926411			
				4·912410			
	2	45	24				
Pol. Dist.	112	24	0	9·850397			
				∴ PZS..	114°	39′	30″
	115	9	24	and OZX..	95	39	0
	109	38	36				
				Bearing of Pt. N.	19	0	30 W.

PROB. 159.

In the preceding example suppose O to be the summit of a mountain whose elevation is 10°: it is required to find the true bearing of the point O. In the preceding calculation to find angle OZX we must substitute the zenith distance (80°) of the point O for the quadrantal arc OZ in last example. We have then three sides given of the triangle OZX to find as before the angle OZX.

Calculation.

ZO..	80°	0′	0″			0·006649
ZX..	79	18	46			0·007594
						4·871860
	0	41	14			4·867153
OX..	95	32	55			
						9·753256
	96	14	9	OZX..	97°	39′
	94	51	41	and PZS..	114	39

Bearing of Mount.. N. 17 0 W.

PROB. 160.

The Solution of this Problem is given in PART I.

Brief explanations of the principal rules in Nautical Astronomy will be found in the Solutions of the following problems:

Prob. 88, 89, 90, 91. To find the latitude by the meridian altitude of a heavenly body.

Prob. 92. The latitude = altitude of pole above the horizon.

Prob. 92. The latitude = half the sum of meridian altitudes above and below pole.

Prob. 105. Latitude by altitude near the meridian.

Prob. 117. Latitude by double altitude.

1st Solution, Inman's Rule; 2nd Solution, Riddle's Rule.

Prob. 103. To find apparent time at the ship or place of observation: (for determining the longitude by chronometer).

Prob. 110. To find the true distance of moon from another heavenly body: (for determining the longitude by lunar observation).

Prob. 103. To find the azimuth of a heavenly body, the altitude being given: (for determining the variation of compass).

Prob. 150. To find the azimuth of a heavenly body, the apparent time at ship being given: (for determining the variation of compass).

Prob. 125. True bearing of heavenly body by amplitude: (for determining the variation of compass).

Prob. A. p. 64. To find the corresponding error in the latitude for a given error in the hour angle of a heavenly body near the meridian.

Prob. B. p. 66. To shew that the corresponding error in hour angle, or apparent time, for a given error in the observed altitude is the least when the bearing of the heavenly body is due west or east.

Fundamental formulæ in Trigonometry: (referred to in the foregoing solutions).

Formulæ for Plane Angles.

$$\text{Sin. } A = \frac{1}{\text{cosec. } A};\ \text{tan. } A = \frac{1}{\text{cot. } A};\ \text{sec. } A = \frac{1}{\text{cos. } A}.$$

(1) $\sin.^2 A + \cos.^2 A = 1$

(2) $\tan. A = \dfrac{\sin. A}{\cos. A}$

(3) $\sin. A = \sin. (180^\circ - A)$

(4) $\cos. A = -\cos. (180^\circ - A)$, &c.

(5) $\sin. (A+B) = \sin. A \cos. B + \cos. A \sin. B$

(6) $\sin. (A-B) = \sin. A \cos. B - \cos. A \sin. B$

(7) $\cos. (A+B) = \cos. A \cos. B - \sin. A \sin. B$

(8) $\cos. (A-B) = \cos. A \cos. B + \sin. A \sin. B$

(9) $\sin. (A+B) + \sin. (A-B) = 2 \sin. A \cos. B$

(10) $\sin. (A+B) - \sin. (A-B) = 2 \cos. A \sin. B$

(11) $\cos. (A+B) + \cos. (A-B) = 2 \cos. A \cos. B$

(12) $\cos. (A+B) - \cos. (A-B) = -2 \sin. A \sin. B$

(13) $\sin. A + \sin B = 2 \sin. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)$

(14) $\sin. A - \sin. B = 2 \cos. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)$

(15) $\cos. A + \cos. B = 2 \cos. \frac{1}{2}(A+B) \cos. \frac{1}{2}(A-B)$

(16) $\cos. A - \cos. B = -2 \sin. \frac{1}{2}(A+B) \sin. \frac{1}{2}(A-B)$

(17) $\sin. 2A = 2 \sin. A \cos. A$

(18) $\cos. 2A = \cos.^2 A - \sin.^2 A = 1 - 2\sin.^2 A = 2\cos.^2 A - 1$

(19) $\text{ver.} A = 2 \sin.^2 \frac{A}{2}$, or $\text{hav.} A = \sin.^2 \frac{A}{2}$

(20) $\frac{\sin. A + \sin. B}{\sin. A - \sin. B} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}$

(21) $\tan. (A+B) = \frac{\tan. A + \tan. B}{1 - \tan. A \tan. B}$

(22) $\tan. (A-B) = \frac{\tan. A - \tan. B}{1 + \tan. A \tan. B}$

(23) $\tan. 2A = \frac{2 \tan. A}{1 - \tan^2 A}$

(24) $\sin. 30^\circ = \frac{1}{2}$; $\sin. 45^\circ = \frac{1}{2}\sqrt{2}$; $\sin. 60^\circ = \frac{1}{2}\sqrt{3}$.

Formulæ for Plane Triangles.

(25) $2bc \cos. A = b^2 + c^2 - a^2$.

(26) $bc\, \text{hav.} A = bc \sin.^2 \frac{A}{2} = \frac{1}{2}(a + \overline{b-c}) \cdot \frac{1}{2}(a - \overline{b-c})$.

(27) $bc \cos.^2 \frac{A}{2} = \frac{1}{2}(b+c+a) \cdot \frac{1}{2}(b+c-a)$.

(28) $2bc \sin A = \sqrt{\left\{(a+\overline{b-c}) \cdot (a-\overline{b-c}) \cdot (b+c+a) \cdot (b+c-a)\right\}}$

(29) $\frac{a}{b} = \frac{\sin. A}{\sin. B}$.

(30) $\frac{a+b}{a-b} = \frac{\tan. \frac{1}{2}(A+B)}{\tan. \frac{1}{2}(A-B)}$.

(31) $a = \frac{\sqrt{4bc \text{ hav. } A}}{\sin. \theta}$ where $\tan. \theta = \frac{\sqrt{4bc \text{ hav. } A}}{b-c}$

(32) $2 \text{ area} = bc \sin. A$.

(33) $\text{Area} = \sqrt{\left\{S\cdot(S-a)\cdot(S-b)\cdot(S-c)\right\}}$

where $S = \frac{1}{2}(a+b+c)$.

Formulæ for Spherical Triangles.

(34) $\cos. A \sin. b \sin. c = \cos. a - \cos. b \cos. c$.

(35) $\frac{\sin. A}{\sin. a} = \frac{\sqrt{(1-\cos.^2 a - \cos.^2 b - \cos.^2 c + 2\cos. a \cos. b \cos. c)}}{\sin. a \sin. b \sin. c}$

(36) $\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}$

(37) $\text{hav. } A = \text{cosec } b \text{ cosec } c \sqrt{\left\{\text{hav.}(a+\overline{b-c})\cdot \text{hav.}(a-\overline{b-c})\right\}}$

or $\sin.^2 \frac{A}{2} = \text{cosec } b \text{ cosec } c \sin. \frac{1}{2}(a+\overline{b-c}) \sin. \frac{1}{2}(a-\overline{b-c})$

(38) $\cos.^2 \frac{A}{2} = \text{cosec } b \text{ cosec } c \sin. \frac{1}{2}(b+c+a) \sin. \frac{1}{2}(b+c-a)$

(39) $\text{ver. } a = \text{ver. } (b \sim c) + \sin. b \sin. c \text{ ver. } A$.

(40) $\cos. a \sin. B \sin. C = \cos. A + \cos. B \cos. C$.

(41) $\cot. A \sin. B = \cot. a \sin. c - \cos. B \cos. c$.

(42) $\tan. \frac{1}{2}(A+B) = \frac{\cos. \frac{1}{2}(a-b)}{\cos. \frac{1}{2}(a+b)}. \cot \frac{1}{2} C$

$\tan. \frac{1}{2}(A-B) = \frac{\sin. \frac{1}{2}(a-b)}{\sin. \frac{1}{2}(a+b)}. \cot. \frac{1}{2} C$.

(43) $\tan. \frac{1}{2}(a+b) = \frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)}. \tan. \frac{1}{2} c$

$\tan. \frac{1}{2}(a-b) = \frac{\sin. \frac{1}{2}(A-B)}{\sin. \frac{1}{2}(A+B)}. \tan. \frac{1}{2} c$.

(44) Napier's Rules for right angled and quadrantal triangles.